Groups and Fields in Neostable Theories Chain: conditions and Definable Envelopes

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Math

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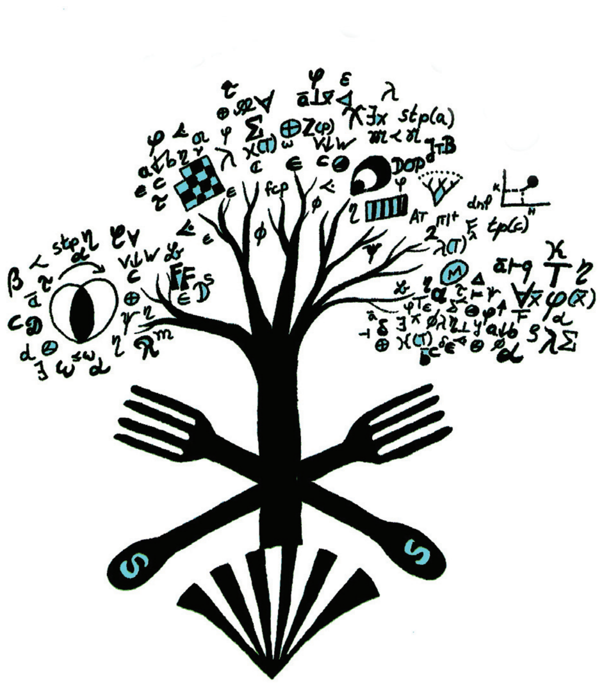
Groupes et Corps dans des Théories Neostables
Condition de Chaîne et Enveloppes Définissables



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Thèse de doctorat

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Thèse de doctorat

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Für meine Eltern

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Introduction en Français

Le théorème de classification de Morley dans les années 60 est à l'origine du développement de la théorie de la stabilité, une branche moderne de la théorie des modèles. Morley introduit alors la notion de théories totalement transcendantes. Ensuite, Shelah considère la classe plus large des théories stables, les théories dont aucune formule n'ordonne un ensemble infini [56]. Les exemples de structures bien connus qui admettent une théorie stable sont celles des corps algébriquement clos, mais aussi des corps séparablement clos ou différentiellement clos, des espaces vectoriels, et des groupes libres. Shelah a introduit de nombreux concepts combinatoires [58] afin de classifier les structures stables et il a été produit au fil des ans de nombreuses techniques et résultats, qui ont trouvé ensuite des applications en géométrie algébrique et en théorie des nombres. De plus, après le travaux de Hrushovski sur les corps pseudo-fini [30], Kim et Pillay ont montré que plusieurs concepts introduits en stabilité peuvent être adaptés et généralisés à la classe des *théories simples* [36, 35, 38]. Celle-ci inclue également les corps aux différences génériques, les corps pseudo-finis, le graphe aléatoire.

L'étude d'une autre classe de théories, les *théories dépendantes*, a été en particulier développée afin de considérer d'un point de vue modèle-théorique d'autres structures mathématiques usuelles, telles que le corps des réels ou le corps des nombres p-adiques, qui ne sont pas simples, mais dépendants. Notons que les théories stables sont exactement celles qui sont à la fois dépendantes et simples. Des travaux ont montré que de nombreuses techniques développées pour les théories stables peuvent être adaptées aux théories dépendantes. Par exemple, Kaplan, Scanlon et Wagner ont généralisé le fait que tout corps infini stable est Artin-Schreier clos à ce contexte plus large [34].

Récemment, plus d'attention a été apportée à une classe de théories englobant les théories simples et les théories dépendantes, la classe des théories qui ne satisfont pas la propriété de l'arbre du deuxième type (*théories NTP_2*). Initialement introduite par Shelah [57], ces théories ont été intensivement étudiées par Chernikov dans sa thèse de doctorat [9]. Dans [11] Chernikov et Kaplan ont montré que la non déviation y a encore de bonnes propriétés, ce qui suggère que d'autres résultats connus pour les théories simples ou dépendantes pourraient être étendus à ce contexte. Chernikov, Kaplan et Simon ont montré que tout corps NTP_2 a au plus un nombre fini extensions d'Artin-Schreier [12]. Certains corps algébriquement clos valués aux différences (Chernikov et Hils [10]) et les corps pseudo réellement clos (Monténégro [48]) sont des corps NTP_2, qui ne sont ni simples ni dépendants.

Dans cette thèse, nous nous explorons certaines propriétés des groupes et corps ayant une théorie dépendante, simple ou seulement NTP_2, c'est-à-dire dans le contexte de la *néostabilité*.

Concernant les groupes, nous nous intéressons à la question de l'existence d'*enveloppes définissables* : étant donnés un groupe G et un sous-groupe H arbitraire qui est de plus commutatif, nilpotent ou résoluble, peut-on trouver un sous-groupe définissable de G contenant H ayant les mêmes propriétés algébriques. Exhiber un ensemble définissable enveloppant un ensemble non définissable donné et satisfaisant des propriétés similaires, est important en théorie des modèles et ses applications, car elle transporte des objets qui sont a priori hors de sa portée, dans la catégorie de ces objets d'étude, celle des ensembles définissables. Au sujet des enveloppes définissables, il y a eu des progrès remarquables au cours des dernières décennies pour des groupes qui satisfont certaines propriétés modèle-théoriques, ainsi que pour des groupes dont les centralisateurs satisfont certaines conditions de chaîne.

Les conditions de chaîne sur les sous-groupes uniformément définissables forment l'un des résultats centraux concernant les groupes définissables dans l'une des théories susmentionnées. Par exemple, dans un groupe stable, toute chaîne descendante d'intersections de sous-groupes uniformément définissables se stabilise après un nombre fini d'étapes. Les groupes dans lesquels les centralisateurs satisfont cette condition de chaîne sont appelés $\mathfrak{M}_c$*-groupes*; les groupes stables sont ainsi des $\mathfrak{M}_c$-groupes. Historiquement, ils ont été d'un grand intérêt à la fois pour les théoriciens des groupes et pour les théoriciens des modèles. Une des propriétés cruciales est que le centre de tout $\mathfrak{M}_c$-groupe est égal au centralisateur d'un nombre fini d'éléments. Bryant montre dans [7] que cette propriété est conservée aux quotients par tout centre itératif. Ce fait a ouvert la voie à d'autres résultats. Par exemple des enveloppes définissables existent pour les sous-groupes abéliens et nilpotents. Alors que les enveloppes définissables pour les sous-groupes abéliens sont faciles à trouver, le cas nilpotent, un résultat de Altinel et Baginski [1], est beaucoup plus complexe. L'une des raisons est le fait qu'un quotient d'un $\mathfrak{M}_c$-groupe par un sous-groupe distingué n'est pas nécessairement un $\mathfrak{M}_c$-groupe. Ceci est un obstacle pour trouver des enveloppes définissables des sous-groupes résolubles de $\mathfrak{M}_c$-groupes, une question qui reste toujours ouverte. Un autre objet d'intérêt est le sous-groupe de Fitting, qui est le groupe engendré par tous les sous-groupes nilpotents distingués. Bien qu'il soit toujours distingué dans le groupe ambiant, et nilpotent pour les groupes finis, il pourrait ne pas être nilpotent pour certains groupes infinis. Bryant a d'abord démontré que le sous-groupe de Fitting de tout $\mathfrak{M}_c$-groupe périodique est nilpotent [7]. En utilisant des techniques modèle-théoriques, Wagner a prouvé la nilpotence du sous-groupe de Fitting d'un groupe stable dans [62] et plus tard Wagner et Derakshan ont généralisé ce résultat aux $\mathfrak{M}_c$-groupes arbitraires [15].

Les sous-groupes uniformément définissables d'un groupe dont la théorie est simple, satisfont une condition de chaîne légèrement plus faible : toute chaîne descendante d'intersections de sous-groupes uniformément définissables chacune ayant indice infini dans son prédécesseur, stabilise après un nombre fini d'étapes. De plus, en utilisant la compacité, on peut trouver pour toute famille donnée de sous-groupes uniformément définissables, des nombres naturels d et n d'une manière que toute chaîne descendante d'intersections de cette famille, chacune ayant indice plus grand que d dans son prédécesseur, a une longueur au plus n. Les groupes pour lesquels les centralisateurs de toutes sections définissables satisfont cette condition de chaîne seront appelés $\widetilde{\mathfrak{M}}_c$-*groupes*. Le fait que tout groupe G définissable dans une théorie simple est un $\widetilde{\mathfrak{M}}_c$-group joue un rôle essentiel dans la preuve de Milliet montrant que tout sous-groupe commutatif de G est contenu dans un sous-groupe définissable fini-par-abélien [47] et que tout sous-groupe résoluble de G est contenu dans un nombre fini de translatés d'un sous-groupe résoluble définissable [46]. Dans le même papier, pour obtenir le résultat correspondant pour les sous-groupes nilpotents de G, il utilise d'autres outils modèle-théoriques provenant des théories simples. Ensuite, Palacín et Wagner ont généralisé ces résultats dans [50] aux groupes type-définissables dans une théorie simple, ce qui leur a permis de montrer dans ce contexte la nilpotence du sous-groupe de Fitting, en utilisant aussi bien la condition de chaîne sur les centralisateurs et des outils développés pour les théories simples.

J'ai isolé les propriétés nécessaires dans les preuves de Palacín et Wagner sur les enveloppes définissables et le sous-groupe de Fitting dans les théories simples, puis j'ai développé une approche purement groupe théorique. Ceci est présenté dans le chapitre 3 qui se consacre à l'étude des *presque centralisateurs* de sous-groupes. Si G est un groupe et A un ensemble de paramètres, nous définissons pour des sous-groupes A-invariants K, H et N, tels que H et K normalisent N, le *presque centralisateur* de H dans K modulo N :

$$\widetilde{C}_K(H/N) = \{k \in K : [H : C_H(k/N)] \text{ est borné}\}.^1$$

On peut penser à cet objet comme l'ensemble des éléments de K qui commutent avec presque tous les éléments de H modulo N. Notons que cet ensemble forme un sous-groupe de K qui est stabilisé par tous les automorphismes qui fixent H, K et N comme ensemble.

Dans le même esprit, on dit qu'un sous-groupe A-invariant H est *presque contenu* dans un autre sous-groupe A-invariant K si l'intersection de H et K a indice borné dans H. On note cette propriété $H \lesssim K$.
De manière analogue, pour des sous-groupes **arbitraires** H et K, on dit que H est *virtuellement contenu* dans K si l'intersection de H et K a indice **fini** dans H.

Nous concentrons notre étude sur la classe des sous-groupes *A-ind-définissables*. C'est une notion modèle théorique qui généralise les sous-groupes type-définissables et qui tombe dans la classe des sous-groupes invariants. Plus précisément, dans cette thèse un sous-groupe A-ind-définissable est l'union d'un système dirigé de sous-groupes A-type-définissables. Étant donné un groupe G, nous disons que les deux sous-groupes H et K de G *normalisent fortement et simultanément* un sous-groupe A-ind-définissable L de G s'il existe un ensemble de sous-groupes A-type-définissables $\{L_\alpha : \alpha \in \Omega\}$ de G chacun normalisé par H et K tel que L est égal à $\bigcup_{\alpha\in\Omega} L_\alpha$.

Les théorèmes ci-dessous résument les résultats principaux du Chapitre 3 (Théorème 3.13, Théorème 3.19, et Théorème 3.24).

[1] voir Notation on page 9 pour clarification

Theorem A. Soit G un groupe et A un ensemble de paramètres. Pour trois sous-groupes A-ind-définissables H, K et L de G , on obtient :

- (*symétrie*) Si N est un sous-groupe de G normalisé par H et K, qui est l'union d'ensembles A-définissables, alors
$$H \lesssim \widetilde{C}_G(K/N) \quad \text{si et seulement si} \quad K \lesssim \widetilde{C}_G(H/N).$$
- (*lemme des trois sous-groupes*) Supposons que H , K et L normalisent fortement et simultanément chacun des autres. Alors,
$$\text{si } H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L)) \text{ et } K \lesssim \widetilde{C}_G(L/\widetilde{C}_G(H)) \text{ alors } L \lesssim \widetilde{C}_G(H/\widetilde{C}_G(K)).$$

Theorem B (généralisation d'un théorème de Neumann)**.** Soit G un groupe et soient H et K deux sous-groupes de G. Nous supposons que

- H normalise K;
- $H \leq \widetilde{C}_G(K)$;
- $K \leq \widetilde{C}_G(H)$, de plus il y a $d \in \omega$ tel que pour tout k dans K la classe de conjugaison k^H a cardinalité au plus d.

Alors, le groupe $[K,H]$ est fini.

En utilisant ces propriétés, nous analysons les $\widetilde{\mathfrak{M}}_c$-groupes. Leur propriété centrale est que le presque centralisateur de chaque sous-groupe est définissable, ce que nous montrons dans la section 3.5. Ensuite, nous généralisons les résultats sur les enveloppes définissables et le sous-groupe de Fitting dans les théories simples aux $\widetilde{\mathfrak{M}}_c$-groupes Pour les énoncer, nous avons besoin d'introduire les versions approximatives des propriétés de commutativité, nilpotence et résolubilité.

Les groupes qui sont *presque abéliens*, c.à.d. les groupes dans lesquels la classe de conjugaison de chaque élément est fini (appelés aussi *FC-groupes*), remontent à Baer et Neumann. De même, Haimo a introduit et étudié les généralisations d'autres propriétés groupe-théoriques classiques. En remplaçant le centre par le FC-centre (qui est le FC-centralisateur du groupe en lui-même) dans la définition des groupes nilpotents, et abélien par presque abélien dans la définition des groupes résolubles, il a introduit la notion de groupes *FC-nilpotents* ou *presque nilpotents* et respectivement de groupes *FC-résolubles* ou *presque résolubles*. Ces objets correspondent à leurs analogues ordinaires pour lesquelles les propriétés restent vraies «à indice fini près».

Nous obtenons des enveloppes définissables dans le contexte des $\widetilde{\mathfrak{M}}_c$-groupes (Propostition 4.17, Théorème 4.19 et Théorème 4.24) et nilpotence de son sous-groupe de Fitting (Théorème 5.9).

Theorem C. Soit G un $\widetilde{\mathfrak{M}}_c$-groupe et H un sous-groupe de G.

1. Si H est presque abélien, il existe un sous-groupe définissable fini-par-abélien de G qui contient H et qui est normalisé par $N_G(H)$.

2. Si H est presque nilpotent de classe n, il existe un sous-groupe définissable nilpotent N de G de classe au plus $2n$ qui contient virtuellement H et qui est normalisé par $N_G(H)$.
   Notamment, si H est distingué dans G, le groupe HN est un sous-groupe définissable, distingué et nilpotent de classe au plus $3n$ qui contiens H.

3. Si H est presque résoluble de classe n, il existe un sous-groupe définissable résoluble N de G de classe au plus $2n$ qui contient virtuellement H et qui est normalisé par $N_G(H)$.
   Notamment, si H est distingué dans G, le groupe HS est un sous-groupe définissable, distingué et résoluble de classe au plus $3n$ qui contiens H.

Theorem D. Le sous-groupe de Fitting d'un $\widetilde{\mathfrak{M}}_c$-groupe est nilpotent et définissable.

En outre, la notion d'un sous-groupe presque nilpotent étant introduite, on peut naturellement considérer le *sous-groupe de Fitting approximatif*, c.à.d. le groupe engendré par tous les sous-groupes presque nilpotents distingués. Nous montrons que le sous-groupe presque Fitting d'un $\widetilde{\mathfrak{M}}_c$-groupe est presque résoluble. (Notons que la première étape pour démontrer la nilpotence du sous-groupe de Fitting est la preuve de résolubilité.)

Pour les groupes dépendants, Shelah a montré dans [59] que tout sous-groupe abélien est contenu dans un sous-groupe abélien définissable (dans une extension saturée) et Aldama a généralisé ce résultat aux sous-groupes nilpotents [14]. Dans le cas résoluble, Aldama ne l'a montré pour l'instant que pour les sous-groupes distingués dans une extension suffisamment saturée.

Il se trouve que les presque centralisateurs et leurs propriétés présentées dans cette thèse sont utiles pour analyser les enveloppes définissables des sous-groupes abéliens, nilpotents ou résolubles distingués d'un groupe NTP_2. Avec Onshuus, nous avons généralisé les résultats sur les enveloppes définissables dans ce contexte. Ceci est présenté dans la section 4.2 et fait partie de l'article [24] qui est accepté pour publication dans «Israel Journal of Mathematics» :

Theorem E. Soit G un groupe définissable dans un théorie NTP_2 et H un sous-groupe de G tel que G est $|H|^+$-saturé.

1. Si H est abélien, alors il existe un sous-groupe définissable fini-par-abélien A de G qui contient H. De plus, si H est distingué dans G, le sous-groupe A peut être choisi également distingué dans G.

2. Si H est nilpotent de classe n, alors il existe un sous-groupe définissable nilpotent N de G de la classe au plus $2n$ qui contient virtuellement H.

De plus, si H est distingué dans G, le groupe N peut être choisi également distingué dans G, et le sous-groupe HN est nilpotent définissable de classe au plus $3n$.

3. Si H est résoluble de classe n et distingué dans G, alors il existe un sous-groupe définissable résoluble distingué S de G de classe au plus $2n$ qui contient virtuellement H. De plus, le sous-groupe HS est définissable, résoluble de classe au plus $3n$ et distingué dans G.

Par analogie avec le presque centralisateur, donné deux sous-groupe A-ind-définissable H et K d'un groupe quelconque, nous définissons une notion de *presque commutateur*, noté $[H,K]^{\sim}$, et nous établissons ses propriétés de base. La definisabilité du presque centralisateur d'un sous-groupe d'un $\widetilde{\mathfrak{M}}_c$-groupes, permet de montrer l'interaction attendue entre le presque commutateur et le presque centralisateur : $[H,K]^{\sim}$ est trivial si et seulement si $H \lesssim \widetilde{C}_G(K)$. Cette correspondance nous permet de prouver le Corollaire 6.18, une version d'un critère de nilpotence de Hall pour des sous-groupes presque nilpotents d'un $\widetilde{\mathfrak{M}}_c$-groupe:

Theorem F. Soit N un sous-groupe A-ind-définisable et distingué d'un $\widetilde{\mathfrak{M}}_c$-groupe G. Si N est presque nilpotent de classe m et $G/[N,N]^{\sim}_A$ est presque nilpotent de classe n, alors G est presque nilpotent de classe au plus $\binom{m+1}{2}n - \binom{n}{2} + 1$.

Les structures dépendantes sont le premier niveau d'une hiérarchie stricte de structures, les structures n-dépendantes (le n-hypergraphe aléatoire est n-dépendant, mais n'est pas $(n-1)$-dépendant). Nous montrons que le groupe équipé d'une forme bilinéaire suivant est 2-dépendant :

Soit $(G, \mathbb{F}_p, 0, +, \cdot)$ la structure où $\mathbb{F}_p$ est le corps fini de cardinalité p, G est le groupe $\bigoplus_\omega \mathbb{F}_p$, la constante 0 est l'élément neutre, + est l'addition dans G et $\cdot$ est la forme bilinéaire $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ de G à $\mathbb{F}_p$.

Cet exemple, dans le cas p est égal à 2 a été étudié par Wagner dans [63, Exemple 4.1.14]. Il montre qu'il est simple et que la composante connexe G_A^0 pour tout ensemble de paramètres A est égal à $\{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$. Par conséquent, il devient de plus en plus petit en élargissant A. Ceci est le premier exemple non combinatoire connu d'une structure 2-dépendante qui n'est pas dépendante et il illustre le fait que la composante connexe, qui est absolu pour un groupe dépendant, peut dépendre des paramètres pour un groupe seulement 2-dépendant.

Rappelons que Kaplan, Scanlon et Wagner ont prouvé que tout corps dépendant infini est Artin-Schreier clos [34] et Duret a montré que tout corps pseudo algébriquement clos (PAC) non séparablement clos ne fait pas partie de la classe des théories dépendants [16]. Nous généralisons ces résultats à la classe des théories n-dépendantes. Ces résultats se trouvent dans le chapitre 7 et font l'objet d'un article [23] accepté pour publication dans le revue «Mathematical Logic Quarterly»:

Theorem G. Tout corps infini n-dépendant est Artin-Schreier clos.

Theorem H. Pour tout nombre naturel n, aucun corps PAC non separablement clos est n-dépendant.

Dans le cas particulier des corps pseudofinis ou, plus généralement, les corps PAC e-libres, le Théorème H est une conséquence d'un résultat de Beyarslan prouvé dans [4] qui dit qu'on peut interpréter les n-hypergraphes dans un tel corps.

La pertinence du fait que des corps PAC non separablement clos ne sont n-dépendants pour aucun n réside dans la conjecture que les (purs) corps supersimples sont précisément les corps PAC parfaits bornés. Ainsi, la conjecture implique que tout (pur) corps supersimple n-dépendant est séparablement clos et donc stable.

Le dernier chapitre est consacré à l'étude des corps gauches. L'une des questions naturelles sur les corps gauches d'un point de vue modèle théorique, est de savoir s'il existe ou non des corps non commutatifs définissables dans des structures ayant certaines propriétés. Par un résultat de Pillay, Scanlon et Wagner, il est connu que tout corps gauche supersimple est commutatif. Plus tard, Milliet a démontré dans [45] que tout corps gauche de caractéristique positive ayant une théorie simple est de dimension finie sur son centre. Cette question est en général ouverte en caractéristique nulle. Sous certaines conditions, nous avons prouvé avec Palacín la commutativité de tout corps gauche ayant une théorie simple (Théorème 8.17) :

Theorem I. Un corps gauche définissable dans une théorie simple avec un générique de poids 1 est commutatif.

En outre, nous avons analysé les corps gauches de fardeaux finis (Théorème 8.20, Corollaire 8.21) :

Theorem J. Tout gauche corps de fardeau n a dimension au plus n sur son centre et de plus sur tout sous corps commutatif définissable et infini.

Sous ces hypothèses, on ne peut espérer améliorer le résultat, car le corps gauche non-commutatif des quaternions est définissable dans les réels et est de fardeau fini.

Introduction

Model theory is a branch of mathematics which concentrates on classifying first order theories of mathematical objects and on studying their definable sets. The classification theorem of Morley in the 1960's and the introduction of *totally transcendental theories* started the development of stability theory. Afterwards, Shelah considered the larger class of *stables theories*, i. e. theories which do not encode a linear order [56]. Well known examples of structures whose theories are stable are algebraically closed fields as well as differentially and separably closed fields, vector spaces over infinite fields and free groups. Shelah introduced numerous combinatorial concepts to study stable structures [58], and over the years, a large catalog of techniques and results has been produced which has found many applications in algebraic geometry and number theory. Moreover, after the work of Hrushovski on pseudo-finite fields [30], Kim and Pillay showed that many of these concepts developed for stable theories can be adapted and generalized to the wider class of *simple theories* [36, 35, 38]. They include generic difference fields, the random graph and pseudo-finite fields.

The aim to study other relevant mathematical examples, such as the reals or the p-adics, which are not simple but whose definable sets have a certain combinatorial property, led to the investigation of the class of *dependent theories*. Note that stable theories are exactly those which are both dependent and simple. Recent effort has shown that many techniques from stability theory can be adapted as well to dependent theories. For example, Kaplan, Scanlon and Wagner generalized the fact that infinite stable fields are Artin-Schreier closed to this wider framework [34].

Recently, an even wider class of theories including both simple and dependent theories has attracted more attention: theories which do not satisfy the tree property of the second kind (*NTP_2 theories*). Originally introduced by Shelah [57], such theories have been intensively studied by Chernikov in his PhD thesis [9]. One important result, proven in a collaboration with Kaplan in [11], is that forking independence is still well-behaved. This gives hope to extend other results known for simple as well as dependent theories to this context. Moreover, Chernikov, Kaplan and Simon have shown that an NTP_2 field has at most finitely many Artin-Schreier extensions [12]. Examples of NTP_2 fields, which are neither simple nor dependent, are certain algebraically closed valued difference field, as shown by Chernikov and Hils [10] and pseudo real closed fields, as shown by Montenegro [48].

The aim of this thesis is to analyze certain properties of groups and fields having a dependent, simple or NTP_2 theory, which we summarize as having a *neo-stable theory*.

Regarding groups, one is for example interested if given a group G and an arbitrary subgroup H which is abelian, nilpotent or solvable, can one find a *definable envelope* of H, that is a definable subgroup of G containing H with the same algebraic properties. In the past decades there has been remarkable progress on groups fulfilling model theoretic properties as well as on groups satisfying certain chain conditions on centralizers which will ensure the existence of definable envelopes. Finding definable sets around non-definable objects admiting similar properties becomes essential as it brings objects outside of the scope of model theory into the category of definable sets. Furthermore, it is interesting not only from a purely model theoretic point of view but also an important tool for applications.

Some of the central results dealing with groups definable in any of the aforementioned theories are chain conditions on uniformly definable subgroups. For example in groups with a stable theory, any descending chain of intersections of uniformly definable subgroups stabilizes after finitely many steps. Groups which satisfy this condition for centralizers are called $\mathfrak{M}_c$*-groups* and hence stable groups are $\mathfrak{M}_c$-groups. Historically, they have been of great interest to both group and model theorists. One of the crucial properties is that the center of any $\mathfrak{M}_c$-group is equal to the centralizer of finitely many elements. Bryant shows in [7] that this property is preserved under taking quotients by any iterated center. This fact paved the way to further results. For example definable envelopes exist for abelian and nilpotent subgroups of $\mathfrak{M}_c$-groups. Whereas the definable envelopes for abelian subgroups are easy to find, the nilpotent case, due to Altinel and Baginski [1], is much more elaborate. One of the reasons is the fact that quotients of $\mathfrak{M}_c$-groups are not necessarily $\mathfrak{M}_c$-groups. This is as well an obstacle to find definable envelopes for solvable subgroups of $\mathfrak{M}_c$-groups, a question which still remains open. Another object of interest is the Fitting subgroup, that is, the group generated by all normal nilpotent subgroups. While it is always normal in the ambient group and nilpotent for finite groups, it might not be nilpotent for infinite groups. In the case of $\mathfrak{M}_c$-groups, Bryant first showed that the Fitting subgroup of any periodic $\mathfrak{M}_c$-group is nilpotent [7]. Using model theoretic techniques, Wagner proved in [62] nilpotency of the Fitting subgroup of any group whose theory is stable and later Wagner together with Derakshan generalized this result to arbitrary $\mathfrak{M}_c$-groups in [15].

In groups with a simple theory a slightly weaker chain condition holds, namely any descending chain of intersections of uniformly definable subgroups each having infinite index in its predecessor, stabilizes after finitely many steps. Moreover, using compactness, one can find natural numbers d and n in a way that any such descending chain of subgroups each having index greater than d in its predecessor has length at most n. We refer to groups which satisfy this chain condition on centralizers in any definable section as $\widetilde{\mathfrak{M}}_c$*-groups*. The fact that any group G definable in a simple theory is an $\widetilde{\mathfrak{M}}_c$-group plays an essential role in the proof of Milliet showing that an arbitrary abelian subgroup of G is contained in a definable finite-by-abelian subgroup [47] and that any solvable subgroup of G is contained in the union of finitely many translates of a definable solvable subgroup [46]. To obtain the corresponding result for nilpotent subgroups of G which can also be found in [46], he uses as well other model theoretic tools coming from simple theories. Moreover, Palacín and Wagner generalized these results in [50] to type-definable groups in a simple theory which enabled them, making use as well of the chain condition on centralizers and model theoretic machinery from simple theories, to show nilpotency of the Fitting subgroup for groups type-definable in a simple theory.

I isolate the necessary results in the proofs of Palacín and Wagner on definable envelopes and nilpotency of the Fitting group in simple theories and give a purely group theoretical approach. This is presented in Chapter 3 which is dedicated to the study of the *almost centralizer* of subgroups. Let G be a group and A be a parameter set. For A-invariant subgroups K, H and N, such that H and K normalize N, we define:

$$\widetilde{C}_K(H/N) = \{k \in K : [H : C_H(k/N)] \text{ is bounded}\}.$$[2]

This group is called the *almost centralizer* of H in K modulo N. One may think of this object as the set of elements of K which commute with almost all elements of H modulo N. Note that this set forms a subgroup of K which is stabilized by all automorphisms which fix H, K and N setwise.

In the same spirit as the almost centralizer, one can define that an A-invariant subgroup H is *almost contained* in another A-invariant subgroup K, i. e. the intersection of H and K has bounded index in H. We denote this by $H \lesssim K$.
Analogously, for **arbitrary** subgroups H and K, we say that H is *virtually contained* in K if the intersection of H and K has **finite** index in H.

We concentrate our study on the class of *A-ind-definable* subgroups. It is a model theoretic notion which generalizes type-definable subgroups and which falls into the class of invariant subgroups. More precisely, in this thesis an A-ind-definable subgroup is the union of a directed system of A-type-definable subgroups. Given a group G, we say that two subgroups H and K of G *simultaneously strongly normalize* an A-ind-definable subgroup L of G if there is a set of A-type-definable subgroups $\{L_\alpha : \alpha \in \Omega\}$ of G each normalized by H and K such that L is equal to $\bigcup_{\alpha\in\Omega} L_\alpha$.

The two theorems below summarize the main results of Chapter 3:

Theorem A. Let G be a group and let A be a parameter set. For H, K and L three A-ind-definable subgroups of G, we obtain the following:

- (*symmetry*) If N is a subgroup of G which is the union of some A-definable sets and normalized by H and K, then
$$H \lesssim \widetilde{C}_G(K/N) \quad \text{if and only if} \quad K \lesssim \widetilde{C}_G(H/N).$$
- (*almost three subgroups lemma*) Suppose H, K and L simultaneously strongly normalize each other.
$$\text{If } H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L)) \text{ and } K \lesssim \widetilde{C}_G(L/\widetilde{C}_G(H)) \text{ then } L \lesssim \widetilde{C}_G(H/\widetilde{C}_G(K)).$$

Theorem B (generalized Neumann theorem)**.** Let G be a group and let H and K be two subgroups of G. Suppose that

[2] see Notation on page 9 for clarification

- H normalizes K;
- $H \leq \mathrm{FC}_G(K)$;
- $K \leq \mathrm{FC}_G(H)$, moreover there is $d \in \omega$ such that for all k in K the set of conjugates k^H has size at most d.

Then the group $[K,H]$ is finite.

Using these properties, we are able to analyze $\widetilde{\mathfrak{M}}_c$-groups. Their crucial property is that the almost centralizer of any invariant subgroup is definable which we prove in Section 3.5. Afterwards, we generalize the results on definable envelopes and the Fitting group in simple theories to $\widetilde{\mathfrak{M}}_c$-groups. To state them, we need to introduce generalized notions of being abelian, nilpotent and solvable.

Groups which are *almost abelian*, i. e. are groups in which the conjugacy class of any element is finite (also referred to as *FC-groups*), date back to Baer and Neumann. Similarly, Haimo introduced and studied generalizations of other classical group theoretic properties. By replacing center by FC-center (that is the FC-centralizer of the group in itself) in the definition of nilpotent groups and abelian by almost abelian in the definition of solvable groups, he introduced the notion of an *FC-nilpotent* or *almost nilpotent* group and respectively an *FC-solvable* or *almost solvable* group. These objects correspond to their ordinary analogs in which the required properties hold “up to finite index”.

We obtain definable envelopes in $\widetilde{\mathfrak{M}}_c$-groups (Proposition 4.17, Theorem 4.19, Theorem 4.24) and nilpotency of their Fitting subgroups (Theorem 5.9):

Theorem C. Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be a subgroup of G. Then the following hold:

1. If H is almost abelian, then there exists a definable finite-by-abelian subgroup of G which contains H and which is normalized by $N_G(H)$.
2. If H is almost nilpotent of class n, then there is a definable nilpotent subgroup N of G of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H. In particular, if H is normal in G, we have that HN is a definable normal nilpotent subgroup of G of class at most $3n$ which contains H.
3. If H is almost solvable of class n, then there exists a definable solvable subgroup S of G of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H. In particular, if H is normal in G, the group HS is a definable normal solvable subgroup of G of class at most $3n$ which contains H.

Theorem D. The Fitting group of any $\widetilde{\mathfrak{M}}_c$-group is nilpotent and definable.

Moreover, given the notion of an almost nilpotent subgroup, one can naturally consider the *almost Fitting subgroup*, namely the group generated by all normal almost nilpotent subgroups. We shall show that the almost Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is almost solvable. Note, that proving solvablity of the Fitting subgroup for $\widetilde{\mathfrak{M}}_c$-groups is the first step to prove its nilpotency.

For groups with a dependent theory, Shelah showed in [59] that any abelian subgroup is contained in a definable abelian subgroup (in a saturated extension) and Aldama generalized this result to nilpotent subgroups [14]. In the solvable case, also due to Aldama, for now, this is only possible if we work in a sufficiently saturated elementary extension in which the solvable subgroup is normal.

Together with Onshuus, we generalized the results on definable envelopes to groups with an NTP_2 theory. This is presented in Section 4.2 and forms part of the article [24] which is accepted for publication in the Israel Journal of Mathematics:

Theorem E. Let G be a group definable in an NTP_2 theory, H be a subgroup of G and suppose that G is $|H|^+$-saturated. Then the following holds:

1. If H is abelian, then there exists a definable finite-by-abelian subgroup of G which contains H.
   Furthermore, if H is normal in G, the definable finite-by-abelian subgroup can be chosen to be normal in G as well.

2. If H is nilpotent of class n, then there exists a definable nilpotent subgroup N of G of class at most $2n$ which virtually contains H.
   Moreover, if H is normal in G, the group N can be chosen to be normal in G as well and HN is a definable nilpotent group of class at most $3n$ which contains H.

3. If H is solvable of class n which is normal in G, then there exists a definable normal solvable subgroup S of G of class at most $2n$ which virtually contains H.
   In particular, the group HS is a definable solvable subgroup of G of class at most $3n$ which contains H.

In analogy to the almost centralizer, given two A-invariant subgroups H and K of some group, we define the *almost commutator* of H and K, denoted by $\tilde{[}H,K\tilde{]}$, and establish its basic properties. As a consequence of the almost centralizer being definable for $\widetilde{\mathfrak{M}}_c$-groups, the almost centralizer and the almost commutator interact in the expected way, namely $\tilde{[}H,K\tilde{]}$ is trivial if and only if $H \lesssim \widetilde{C}_G(K)$. This correspondence enables us to prove Corollary 6.18, a version of Hall's nilpotency criteria for almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups:

Theorem F. Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. If N is almost nilpotent of class m and $G/\tilde{[}N,N\tilde{]}_A$ is almost nilpotent of class n then G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$.

Dependent structures are the first level of a strict hierarchy of structures, called n-dependent, for which the random n-hypergraph is n-dependent but it is not $(n-1)$-dependent. We shall show that the following non dependent group equipped with a bilinear form is 2-dependent:

Let $(G, \mathbb{F}_p, 0, +, \cdot)$ be the structure where $\mathbb{F}_p$ is the finite field with p elements, G is the group $\bigoplus_\omega \mathbb{F}_p$, 0 is the neutral element, + is addition in G, and $\cdot$ is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to $\mathbb{F}_p$.

This example in the case p equals 2 has been studied by Wagner in [63, Example 4.1.14]. He shows that it is simple and that the connected component G_A^0 for any parameter set A is equal to $\{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$. Hence, it is getting smaller and smaller while enlarging A. This is the first known non combinatorial example of such a structure which is not dependent and it illustrates that the connected component, which is absolute in any dependent group, might depend on parameters in 2-dependent groups.

Kaplan, Scanlon and Wagner proved that any infinite dependent field is Artin-Schreier closed [34] and Duret showed in [16] that any non separably closed pseudo algebraically closed (PAC) field does not belong to the class of dependent theories. We generalized these results to the wider class of n-dependent theories. This can be found in Chapter 7 of this thesis and forms part of the article [23] which is accepted for publication in the Mathematical Logic Quarterly:

Theorem G. Any infinite n-dependent field is Artin-Schreier closed.

Theorem H. For any natural number n, any non separably closed PAC field is not n-dependent.

In the special case of pseudofinite fields or, more generally, e-free PAC fields, Theorem H is a consequence of a result of Beyarslan proved in [4], namely that one can interpret the n-hypergraph in any such field. The relevance of the fact that non separably closed PAC fields are not n-dependent for any n lies in the conjecture that (pure) supersimple fields are precisely the bounded perfect PAC fields. Thus, the conjecture implies that any pure n-dependent supersimple field is separably closed and therefore stable.

The last chapter is dedicated to the study of division rings. One of the natural questions to ask about division rings from a model-theoretic point of view, is whether or not non-commutative division rings definable in structures with certain properties exist. Due to Pillay, Scanlon and Wagner it is known that any supersimple division ring is commutative. Later, Milliet showed in [45] that any division ring of positive characteristic with a simple theory is finite dimensional over its center. But less is known for division rings of characteristic zero. Together with Palacín, we have proved commutativity of division rings with a simple theory under certain conditions which can be found as Theorem 8.17:

Theorem I. A definable division ring in a simple theory with a generic type of weight 1 is a field.

Moreover, we analyzed division rings of finite burden (Theorem 8.20, Corollary 8.21):

Theorem J. Any infinite division ring of burden n has dimension at most n over its center and moreover over any infinite definable subfield.

In this setup, one cannot hope to improve the statement, as the non-commutative division ring of the quaternions is definable in the reals and thus has finite burden.

1 Preliminaries

1.1 Classification theory

In this section we introduce the different model theoretic frameworks considered within this thesis.

There are multiple equivalent definitions for the different model theoretical classes we introduce. Some of them have their origins in forbidden combinatorial configurations of definable sets, others are linked to the space of types or the behavior of non-forking. For each theory, we first give the combinatorial definition as it provides the right setup to prove the different chain conditions of uniformly definable subgroups which play an essential role in this thesis. Afterwards, we point out some of the equivalent characterizations.

Stability theory has its origins in Morley's proof of Łoś conjecture in the 60's:

Fact 1.1 (Morley's theorem)**.** *[27, Theorem 12.2.1] Let T be a first-order theory in a countable language which is categorical for an uncountable cardinal. Then it is categorical in all uncountable cardinalities.*

Later on, Morley discovered that such theories must be ω-stable, namely for any countable set A, the set of complete types over A is countable. Afterwards, Shelah took this step further and tried to describe, given a complete first order theory T, the function which maps a cardinal κ to the number of models of T of size κ. Morley conjectured that for any complete theory, this function is nondecreasing for uncountable cardinals. The main philosophical idea to analyze these functions was to isolate certain combinatorial patterns such that any theory which "encodes" such a pattern has a maximal number of models. For theories failing to encode this pattern, one takes a closer look at their space of types. Shelah showed that a meaningful dividing line lies in between theories with a small space of types, which we call stable theories (introduced below), and unstable theories. This marked the beginning of stability theory. The techniques he developed allowed him to affirm Morley's conjecture and further work by Hart, Hrushovski and Laskowski [22] led to a complete description of the possibilities for the aforementioned function.

So let us now give one of the precise definitions of a stable theory:

Definition 1.2 (stable theories)**.** Let T be a theory. A formula $\phi(\bar{x};\bar{y})$ has the *order property* if there are a sequences of tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_j : j \in \omega)$ in some model $\mathcal{M}$ of T such that

$$\mathcal{M} \models \phi(\bar{a}_i;\bar{b}_j) \text{ if and only if } i < j.$$

A formula is called *stable*, if it does not have the order property and a theory T is *stable* if any formula is stable.

Equivalently a theory is stable if for some infinite cardinal κ and any parameter set A of size κ, the space of complete types over A has size at most κ.

A third characterization of stable theories is that any type is definable, i. e. for any complete type $p(x)$ over a model $\mathcal{M}$ and any formula $\phi(x;\bar{y})$ there is a formula $\psi_{p,\phi}(\bar{y})$ with parameters in $\mathcal{M}$ such that for any $\bar{b}$ in $\mathcal{M}$

$$\phi(x;\bar{b}) \in p \quad \Leftrightarrow \quad \mathcal{M} \models \psi_{p,\phi}(\bar{b}).$$

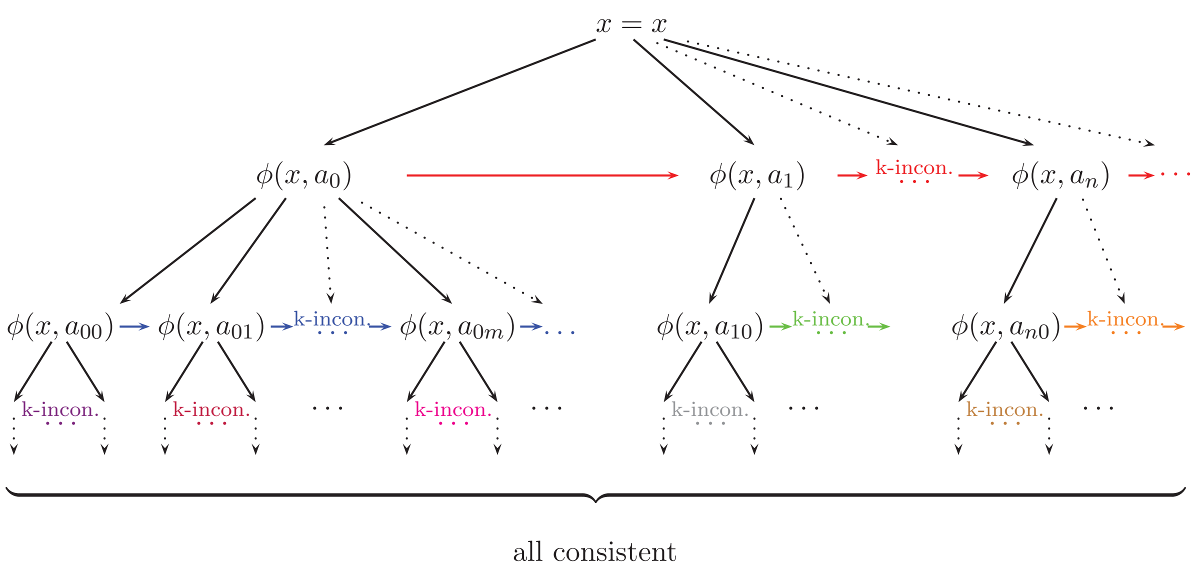
Important examples of stable theories are algebraically closed fields, separably closed fields, differentially closed fields, vector spaces over infinite fields, free groups and planar graphs.

A well know generalization of stable theories are simple theories. They were introduced by Shelah in [57] while studying the saturation spectrum. Their importance came to light through the work of Kim and Pillay [36, 35, 38], in which they generalized purely model theoretic properties of stable theories to this wider context, as well as Hrushovski's results on pseudofinite fields [30] where he proved, in particular, that their theory is simple.

Definition 1.3 (simple theories)**.** Let T be a theory. A formula $\phi(\bar{x};\bar{y})$ has the *tree property* if there exists a parameter set $\{\bar{a}_\mu : \mu \in \omega^{<\omega}\}$ and $k \in \omega$ such that

- $\{\phi(\bar{x};\bar{a}_{\mu^\frown i}) : i < \omega\}$ is k-inconsistent for any $\mu \in \omega^{<\omega}$;
- $\{\phi(\bar{x};\bar{a}_{s\restriction n}) : n \in \omega\}$ is consistent for any $s \in \omega^\omega$.

A theory is *simple* if no formula has the tree property.



The class of simple theories includes, in addition to stable structures, pseudofinite fields, generic difference fields and the random graph. It can also be characterized using forking independence:

Definition 1.4. Let k be a natural number. A formula $\varphi(x,a)$ *k-divides* over a set A if there is a sequence $(a_i : i \in \omega)$ of realizations of $\mathrm{tp}(a/A)$ such that $\{\varphi(x,a_i) : i \in \omega\}$ is k-inconsistent. A partial type $\pi(x)$ divides over A if there is some formula $\varphi(x,a)$ and natural number k such that $\phi(x,a)$ k-divides over A and $\pi(x) \vdash \varphi(x,a)$.
A formula $\varphi(x,a)$ *forks* over A if there are $\psi_i(x,b_i)$ and $k_i \in \omega$ for $i < n$ such that

$$\varphi(x,a) \vdash \bigvee_{i<n} \psi_i(x,b_i)$$

and for each $i < n$, the formula $\psi_i(x,b_i)$ k_i-divides over A. We say that a partial type $\pi(x)$ forks over A if there is some formula $\varphi(x,a)$ which forks over A and $\pi(x) \vdash \varphi(x,a)$.
A type p over B which does not fork over a subset A of B is called a *non-forking extension* of $p \restriction A$.

In an arbitrary first-order theory forking and dividing may not agree, but Kim showed in [35] that in simple theories these two notions coincide.

Using the combinatorial notion of forking, we define *forking independence* as a ternary relation denoted by $\mathop{\smile\!\!\!\!\!\!|}$ among small sets of the monster model $\mathscr{C}$ such that for all small subsets A, B, C of $\mathscr{C}$, we have that

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \Leftrightarrow \text{ for any enumeration } \bar{a} \text{ of } A, \mathrm{tp}(\bar{a}/BC) \text{ does not fork over } C.$$

Kim and Pillay showed that forking independence satisfies the following list of axioms which initiated the study of abstract independence relations [35, 37].

Fact 1.5. *Let T be a simple theory, $\mathscr{C}$ be a monster model of T and $\mathcal{M}$ be an arbitrary model of T. For any small subsets A, B, C and D of $\mathscr{C}$ and a, a_1, a_2, b_1 and b_2 finite tuples, forking independence $\mathop{\smile\!\!\!\!\!\!|}$ satisfies:*

1. *Invariance under $\mathrm{Aut}(\mathscr{C})$: if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $f \in \mathrm{Aut}(\mathscr{C})$, then $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$.*
2. *Finite character: $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$, $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$.*
3. *Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $B \mathop{\smile\!\!\!\!\!\!|}_C A$.*
4. *Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{CB} D$.*
5. *Extension: If $a \mathop{\smile\!\!\!\!\!\!|}_C B$ and $D \supseteq B$, then there is some $a' \equiv_{CB} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_C D$.*
6. *Local character: There is some $E \subseteq B$ with $|E| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\!|}_E B$.*
7. *Strictness: If $A \mathop{\smile\!\!\!\!\!\!|}_C A$, then $A \subseteq \mathrm{acl}(C)$.*
8. *Independence Theorem over models: if $a_1 \equiv_M a_2$, $a_i \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} b_i$ for $i = 1,2$ and $b_1 \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} b_2$, then there is some $a \equiv_{\mathcal{M}b_i} a_i$ for $i = 1,2$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} b_1 b_2$.*

Conversely, any theory that admits an abstract ternary relation which satisfies 1 - 8 is simple and this independence relation coincides with forking independence.

Many results for simple theories whose proof makes use of this forking calculus turned out to be true for o-minimal structures. These are ordered structures in which every definable subset is the finite union of points and intervals. They lie outside the class of simple theories. For such theories one can find an independence relation, which differs from forking independence, satisfying 1-7. This led to the study of the wider class of *rosy theories*, which are exactly those which admit an abstract ternary independence relation satisfying 1-7. These include all simple as well as all o-minimal structures.

Recently another generalization of stable theories, namely dependent theories, has attracted much interest. The original definition, due to Shelah is the following:

Definition 1.6 (dependent theories)**.** Let T be a theory. A formula $\phi(\bar{x};\bar{y})$ has the *independence property* if there are tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_I : I \subseteq \omega)$ in some model $\mathcal{M}$ of T such that

$$\mathcal{M} \models \phi(\bar{a}_i;\bar{b}_I) \text{ if and only if } i \in I$$

A formula is called *dependent* if it does not have the independence property, and a theory is *dependent* if any formula is dependent.

Examples of dependent theories are ordered abelian groups, the reals, the p-adics and algebraically closed valued fields.

Dependent theories are often referred to as NIP theories in the literature. Intuitively they are theories in which one cannot define all subsets of an infinite set by instances of a fixed formula.

An equivalent and very useful characterization of dependent theories is given by indiscernible sequences. One can show that a theory T is dependent if in no model M of T one can find an indiscernible sequence $(\bar{a}_i)_{i\in\omega}$ and a formula $\phi(\bar{x};\bar{b})$ such that $\phi(\bar{a}_i;\bar{b})$ holds in M if and only if i is odd. This can be found in [61].

Another description of dependent theories via VC-dimensions of families of definable sets was given by Laskowski in [43]:

Definition 1.7. Let X be a set, $\mathcal{S}$ be a family of subsets of X, and let A be a subset of X.

- We say that $\mathcal{S}$ *shatters* A if for every $A' \subseteq A$ there is a set S in $\mathcal{S}$ such that $S \cap A = A'$.
- The family $\mathcal{S}$ has *VC-dimension* at n, denoted by $\mathrm{VC}(\mathcal{S}) = n$, if there is no subset of X of cardinality $n+1$ which is shattered by $\mathcal{S}$ but there is a subset of X of size n that is shattered by $\mathcal{S}$.
  If for each n we can find a subset of X of cardinality n that is shattered by $\mathcal{S}$, then we say that $\mathcal{S}$ has infinite VC-dimension, denoted by $\mathrm{VC}(\mathcal{S}) = \infty$.

Laskowski shows that a theory is dependent if and only if for any formula ϕ, any family of ϕ-definable sets has finite VC-dimension.

There is a natural generalization of dependent theories to higher dimensions, namely those in which one cannot define all subsets of ω^n for some natural number n. Formally we obtain the following definition given by Shelah in [60, Definition 2.4].

Definition 1.8 (n-dependent theories)**.** Let T be a theory and n be a natural number. We say that a formula $\psi(\bar{y}_0,\dots,\bar{y}_{n-1};\bar{x})$ in T has the *n-independence property* (IP_n) if there exists some parameters $(\bar{a}_i^j : i \in \omega, j \in n)$ and $(\bar{b}_I : I \subset \omega^n)$ in some model $\mathcal{M}$ of T such that

$$\mathcal{M} \models \psi(\bar{a}^0_{i_0},\dots,\bar{a}^{n-1}_{i_{n-1}},\bar{b}_I) \text{ if and only if } (i_0,\dots,i_{n-1}) \in I.$$

A formula is said to be *n-dependent* if it does not have IP_n. A theory is n-dependent if every formula is n-dependent.

It is easy to see that any theory with the $(n+1)$-independence property has as well the n-independence property. On the other hand, the classes of n-dependent theories form a proper hierarchy as for any natural number n the random $(n+1)$-hypergraph is $(n+1)$-dependent but has the n-independence property [13, Example 2.2.2]. Additionally, since all random hypergraphs are simple, the previous examples show that there are theories which are simple and n-dependent but which are not dependent.

The facts below are useful in order to prove that a theory is n-dependent as they reduce the complexity of the formulas one has to consider. The first one is stated as [60, Remark 2.5] and afterwards proved in detail as [13, Theorem 6.4].

Fact 1.9. *A theory T is n-dependent if and only if every formula $\phi(\bar{y}_0,\dots,\bar{y}_{n-1};x)$ with $|x|=1$ is n-dependent.*

Fact 1.10. *[13, Corollary 3.15] Let $\phi(\bar{y}_0,\dots,\bar{y}_{n-1};\bar{x})$ and $\psi(\bar{y}_0,\dots,\bar{y}_{n-1};\bar{x})$ be n-dependent formulas. Then so are $\neg\phi$, $\phi\wedge\psi$ and $\phi\vee\psi$.*

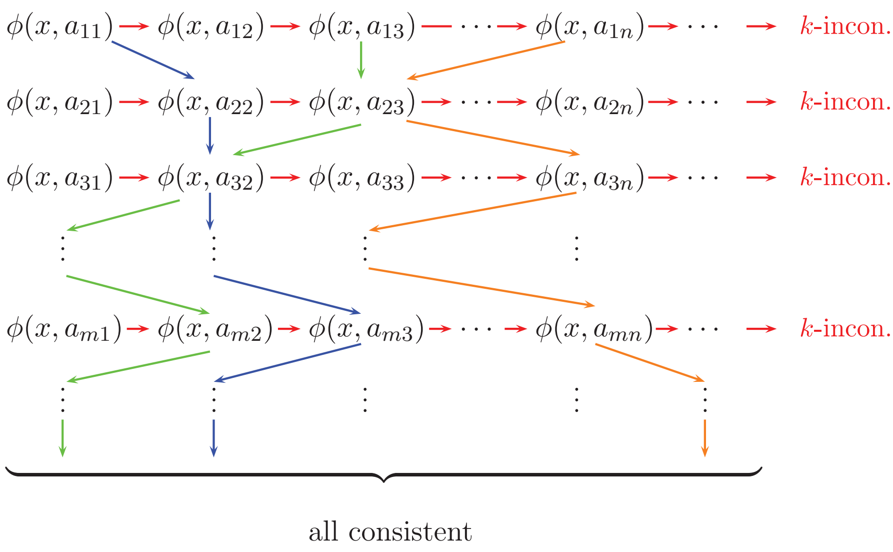
Remark 1.11. Note that a formula with at most n free variables cannot witness the n-independence property. Thus, from the previous fact it is easy to deduce that the random n-hypergraph is n-dependent. In fact, more generally any theory in which any formula of more than n free variables is a boolean combination of formulas with at most n free variables is n-dependent.

Last but not least, we introduce the class of theories without the tree property of the second kind.

Definition 1.12. A theory has the *tree property of the second kind* (referred to as TP_2) if there exists a formula $\psi(\bar{x};\bar{y})$, an array of parameters $(\bar{a}_{i,j} : i,j \in \omega)$, and $k \in \omega$ such that:

- $\{\psi(\bar{x};\bar{a}_{i,j}) : j \in \omega\}$ is k-inconsistent for every $i \in \omega$;
- $\{\psi(\bar{x};\bar{a}_{i,f(i)}) : i \in \omega\}$ is consistent for every $f : \omega \to \omega$.

A theory is called *NTP_2* if it does not have the TP_2.



Chernikov and Kaplan have shown that forking and dividing coincide over models for NTP_2 theories [11].

Remark 1.13. By compactness, having the tree property of the second kind is equivalent to the following finitary version:

A theory has TP_2 if there exists a formula $\psi(\bar{x};\bar{y})$ and a natural number k such that for any natural numbers n we can find an array of parameters $(\bar{a}_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq n)$ satisfying the following properties:

- $\{\psi(\bar{x};\bar{a}_{i,j}) : j \leq n\}$ is k-inconsistent for every i;
- $\{\psi(\bar{x};\bar{a}_{i,f(i)}) : i \leq n\}$ is consistent for every $f : \{1,\ldots,n\} \to \{1,\ldots,n\}$.

Note that the triangle-free random graph has TP_2 [9, Example 3.4.13]. On the other hand as it is a Fraïssé limit in a relational language, it eliminates quantifiers [27, Theorem 7.4.1]. Hence, with the same argument as for the random graph, the triangle-free random graph is 2-dependent. More on the triangle-free random graph can be found in [25].

In general, we shall say that a structure has one of the above properties if its theory does. Moreover, we say stable (simple, dependent, ...) group (field, division ring) for any group (field, division ring) whose theory is stable (simple, dependent, ...).

1.2 Model theory of groups and fields

For this section we fix a theory T and a model $\mathcal{M}$ of T and let A_0 be a parameter set.

Definition 1.14. An *A_0-definable group* G in the theory T is given by a formula $\phi(\bar{x})$ over A_0, a definable binary function $\psi(\bar{x},\bar{y},\bar{z})$ over A_0 with $|\bar{x}| = |\bar{y}| = |\bar{z}|$ and an element e satisfying $\phi(\bar{x})$ such that $\psi(\bar{x},\bar{y},\bar{z})$ defines a group structure on $G = \phi(\mathcal{M})$ with neutral element e.

An A_0-*type-definable group* G in the theory T is given by a type $\pi(\bar{x})$ over A_0, a definable binary function $\psi(\bar{x},\bar{y},\bar{z})$ over A_0 with $|\bar{x}| = |\bar{y}| = |\bar{z}|$ and an element e satisfying $\pi(\bar{x})$ such that $\psi(\bar{x},\bar{y},\bar{z})$ defines a group structure on $G = \pi(\mathcal{N})$ with neutral element e for any elementary extension $\mathcal{N}$ of $\mathcal{M}$.

We might omit the parameter set and just say definable or type-definable group. By the previous definition, an A_0definable group is given by a formula over A_0 and hence any automorphism of $\mathcal{M}$ fixing A_0 induces an automorphism of the definable group. Moreover, observe that G has a natural interpretation in elementary extension of $\mathcal{M}$, namely $\psi(\bar{x},\bar{y},\bar{z})$ remains a binary function which defines a multiplication on the set of elements satisfying $\phi(x)$. Thus we may consider G in any elementary extension of the given model $\mathcal{M}$. So for some cardinal κ, a κ-saturated or κ-homogeneous extension of G is the interpretation of G in an κ-saturated or κ-homogeneous extension of the model in which the group is defined. Sometimes we refer to a definable group seen in an κ-saturated (respectively κ-homogeneous) model of the theory as an κ-saturated (respectively κ-homogeneous) group.

Remark 1.15. Definability for fields or division rings is given analogously, and we use the same terminology.

Definition 1.16. Let G be a A_0-definable group in the theory T, H be a subgroup of G and A be a parameter set containing A_0. We say that H is

- A-*definable* if there is a formula $\psi(\bar{x})$ with parameters in A such that $H = \psi(G)$.
- A-*type-definable* if there is a type $\pi(\bar{x})$ over A such that $H = \pi(G)$.
- A-*invariant* if H is fixed by all automorphisms of $\mathcal{M}$ which fixes A point-wise.

If G is a type-definable group, we say that a subgroup H is *relatively definable* if there is a formula $\psi(\bar{x})$ such that H equals $\psi(G)$.

As definable subgroups are also definable groups they have a natural interpretation in any elementary extension of $\mathcal{M}$ as mentioned above. An A-type-definable subgroup has obviously a setwise analog in any elementary extension of $\mathcal{M}$ as well. However, observe that if the model $\mathcal{M}$ is not $|A|^+$-saturated, an A-type-definable group might coincide with the trivial element in $\mathcal{M}$ but not necessarily in all elementary extensions. Thus, while working with A-type-definable subgroups, we want to place ourselves in an $|A|^+$-saturated model. Then, again any interpretation of an A-type-definable subgroup in an elementary extension forms a group. Last but not least, an A-invariant subgroup of any $|A|^+$-saturated and $|A|^+$-homogeneous group is setwise the union of type-definable sets. Hence an A-invariant subgroup H of G, as a set, has a canonical interpretation in any elementary extension $\mathcal{G}$ of G (the set of realizations in $\mathcal{G}$ of the family of types which define H). By saturation, this set forms again an A-invariant subgroup of $\mathcal{G}$ and we denote it by $H(\mathcal{G})$.

So from now on, an A-invariant subgroup of G, is the trace of an A-invariant subgroup of an $|A|^+$-saturated and $|A|^+$-homogeneous extension of G. Thus, they have a canonical interpretation in any model of the theory.

Remark 1.17. Let H and K be two A-invariant subgroups of an $|A|^+$-saturated and $|A|^+$-homogeneous group G and $\mathcal{G}$ be an elementary extension of G. By saturation and homogeneity, the normalizer of H in K is the trace in K of the normalizer of $H(\mathcal{G})$ in $K(\mathcal{G})$. Moreover, if H is normalized by K, the group $H(\mathcal{G})$ is normalized by $K(\mathcal{G})$.

Now we want to analyze the index of a (relatively) definable or type-definable H of a given definable or type-definable group G as well as the index of an A-invariant subgroup H in another A-invariant subgroup K.
If H and G are both A-definable and the index of H in G is finite in some model containing A, then this index does not depend on the model we chose. If not, it is infinite and always at least as big as the saturation of the chosen model we are working in.
If G is A-type-definable, H is A-relatively definable and we work in an $|A|^+$-saturated model $\mathcal{M}$, then either the index of H in G is finite and its value does not depend on $\mathcal{M}$ or the index is infinite and it may be as large as we want if we pass to a suitable elementary extension of $\mathcal{M}$.
The case where G as well as H are A-type-definable, we have two options regarding the index of H in G: it is either *bounded*, i. e. it does not grow bigger than a certain cardinal while enlarging the model, or for any given cardinal κ, we can find a model such that the index is larger than κ. Then we say that the index is *unbounded*. Note that if the index is bounded it is indeed bounded by $\left(2^{|T(A)|}\right)^+$.
The same dichotomy holds for two A-invariant subgroups H and K of any A-type-definable group G.

Definition 1.18. Let G be a definable group and A be a parameter set.

- G is *connected* if it has no proper definable subgroups of finite index.
- The *A-connected component* of G, denoted by G^0_A, is the intersection of all A-definable subgroups of finite index.
- The *A-type-connected component* of G, denoted by G^{00}_A, is the intersection of all A-type-definable subgroups of bounded index.
- The *A-∞-connected component* of G, denoted by G^∞_A, is the intersection of all A-invariant subgroups of bounded index.

We say that the *connected component exists* if for all small parameter sets B, we have that $G^0_B = G^0_\emptyset$ and denote this subgroup by G^0. Similarly, we define the existence of the type-connected component G^{00} and the ∞-connected component G^∞.

The terminology originates in algebraic geometry. The model theoretic connected component exists in any group definable in an algebraic closed field. These are exactly the algebraic groups as pointed out in the introduction. Furthermore, they have

a connected component when seen as an algebraic variety. So, in fact the connected component which contains the neutral element of the variety coincides with the model theoretic connected component of G. Moreover, for any group G definable in a stable theory, the model theoretic connected component exists. Additionally, in such a group G we have that G^0 equals G^{00}, as any type-definable group can be written as a intersection of definable groups.
For groups definable in dependent theories, the three connected component exist.
In other frameworks such as simple and NTP_2 none of the connected components exists necessarily.

Notation

Let G be a group and H, K and L be three subgroups and g be an element of G. By $[H,K]$ we denote the subgroup generated by all commutators $[h,k] = h^{-1}k^{-1}hk$ with h in H and k in K.

Second, by $[H,K,L]$ we denote the group $[[H,K],L]$.

Third, we may inductively define $[H,_n g]$ and $H^{(n)}$ for any natural number n:

$$[H,_1 g] = [H,g] \quad \text{and} \quad [H,_{n+1} g] = [[H,_n g],g] \text{ for n >0,}$$

$$H^{(1)} = H \quad \text{and} \quad H^{(n+1)} = [H^{(n)},H^{(n)}] \text{ for n >0.}$$

Moreover, if K is normalized by H, we set H/K to be $H/H \cap K$.

If g is an element of $N_G(N)$, we let $C_H(g/N)$ the subgroup of H which contains all elements h in H such that $hg \cdot N = gh \cdot N$.

We say that H contains a subgroup K up to finite index, if $[K : H \cap K]$ is finite.

For two element a and b in G, we write a^b for $b^{-1}ab$.

Part I

Groupes

2 Chain conditions on subgroups

2.1 Uniformly definable subgroups

For this section, we fix a group G definable in some model $\mathcal{M}$ of a theory T and let M be the underlying set of $\mathcal{M}$.

We say that a family of subgroups of G is *uniformly definable* if any member of this collection is of the form $\phi(G;\bar{a})$ for a fixed formula $\phi(x;\bar{y})$ and some tuple $\bar{a}$ in M. The ordinary *descending chain condition* on intersection of uniformly definable subgroups states that an arbitrary intersections of such subgroups is equal to a finite subintersection. This condition does not necessary hold in groups definable outside of stable theories but one can find suitable modifications of such a chain condition in more general frameworks. These have been a key tool for finding definable envelopes of arbitrary subgroups which are abelian, nilpotent or solvable as well as analyzing algebraic properties of the given structures. In this chapter we recall the chain conditions which are known for families of uniformly definable subgroups in groups definable in stable, dependent or simple theories as well as generalize these to wider classes of theories. To do so, let $\phi(x;\bar{y})$ be a formula such that for any tuple $\bar{b}$ in M the set

$$H_{\bar{b}} = \phi(G;\bar{b})$$

defines a subgroup of G.

In the case that T is a stable theory, we obtain the ordinary descending chain condition on intersections of uniformly definable subgroups:

Fact 2.1 (ICC). *If T is a stable theory, then there is a natural number n_ϕ such that for every parameter set $A \subset M^{|\bar{y}|}$ there exists a finite subset A_0 of A of size at most n_ϕ such that the intersection $\bigcap_{\bar{a}\in A} H_{\bar{a}}$ is equal to the finite subintersection $\bigcap_{\bar{a}\in A_0} H_{\bar{a}}$.*

Using the ICC, it is easy to see that any abelian subgroup of any stable group is contained in a definable abelian subgroup. Furthermore, given an arbitrary nilpotent or solvable subgroup, Poizat proved that one can find a definable nilpotent or respectively solvable group of the same nilpotency class or respectively derived length which contains the given group [52].
Another result treated by Wagner in [62] using the ICC is that the Fitting subgroup, namely the subgroup generated by all normal nilpotent subgroups, is nilpotent. Finally, Scanlon used this chain condition to prove that an infinite field of positive characteristic which is definable in a stable theory is Artin-Schreier closed [54].

In the case of simple theories one obtains a chain condition up to finite index:

Fact 2.2 (ICC^0)**.** *[63, Theorem 4.2.12] If T is a simple theory, then there are natural numbers n_ϕ and d_ϕ such that for every parameter set $A \subset M^{|\bar{y}|}$, there exists a finite subset A_0 of size at most n_ϕ such that for all $\bar{b}$ in A, the index of $\bigcap_{\bar{a}\in A_0\cup\{\bar{b}\}} H_{\bar{a}}$ in $\bigcap_{\bar{a}\in I} H_{\bar{a}}$ is less than d_ϕ.*

Proof. Suppose there exists a set of parameters $\{\bar{b}_i : i \in \omega\}$ such that

$$H_{\bar{b}_0} > H_{\bar{b}_0} \cap H_{\bar{b}_1} > \dots > \bigcap_{i\le n} H_{\bar{b}_i} > \dots$$

is an infinite descending chain of intersection of subgroups each having infinite index in its predecessor. Thus, for every natural number n we may choose a set of representatives $\{h_{n,j} : j \in \omega\}$ of different cosets of $\bigcap_{i\le n} H_{\bar{b}_i}$ in $\bigcap_{i<n} H_{\bar{b}_i}$. Now, for any finite sequence $\eta = (i_0, \dots, i_n)$ of natural numbers, we let $\bar{a}_\eta = (\bar{b}_n, \prod_{j=0}^{n} h_{j,i_j})$ and consider the formula

$$\theta(z; \bar{y}, v) = \exists w \ (\phi(w; \bar{y}) \ \wedge \ z = v \cdot w).$$

So $\theta(z; \bar{a}_\eta)$ defines the coset $\prod_{j=0}^{n} h_{j,i_j} \cdot H_{\bar{b}_n}$. Thus for finite sequence of natural numbers η say of length n and two distinct natural numbers i and j, we have that $\theta(z; \bar{a}_{\eta^\frown i})$ and $\theta(z; \bar{a}_{\eta^\frown j})$ define two distinct cosets of $H_{\bar{b}_n}$ in G and thus they are inconsistent. On the other hand, the finite conjunction $\bigwedge_{i\le n} \theta(z; \bar{a}_{\eta\restriction i})$ for some $\eta = (i_0, \dots, i_n) \in \omega^n$ is satisfied by any element in the set $\prod_{j=0}^{n} h_{j,i_j} \cdot \bigcap_{i=0}^{n} H_{\bar{b}_n}$ which is nonempty. Thus, by compactness we have that for any infinite sequence s of natural numbers, the set $\{\theta(z; \bar{a}_{s\restriction i}) : i \in \omega\}$ is consistent. This yields a contradiction to T being a simple theory and the statement is established. □

Using the ICC^0 in groups with a simple theory, one has to slightly adapt the notion of definable envelopes of subgroups to obtain a result. In fact, Milliet proved in [47] that any abelian subgroup of a group with a simple theory is contained in a definable finite-by-abelian subgroup. Moreover, in [46] he showed that for any nilpotent or solvable subgroup of class n one can find a definable nilpotent or respectively solvable subgroup of class at most $2n$ which contains the given group up to finite index. Using the existence of definable envelopes up to finite index, Palacín and Wagner proved nilpotency of the Fitting subgroup for any group type-definable in a simple theory [50]. Our aim in Section 4.3 is to generalize these results to groups in which every definable section satisfies the ICC^0 merely for centralizers.

One can reformulate the proof of the ICC^0 in simple theories in terms of forking independence and the corresponding D-rank (see section 4.2 in [63]). Doing so, one realizes that the ICC^0 holds in any theory that admits an independence relation satisfying all properties of forking independence, except possibly the Independence Theorem. Thus in the wider class of rosy theories, one obtains the same result.

Fact 2.3. *If T is a rosy theory, then there are natural numbers n_ϕ and d_ϕ such that for every parameter set $A \subset M^{|\bar{y}|}$, there exists a finite subset A_0 of size at most n_ϕ such that for all $\bar{b}$ in A, the index of $\bigcap_{\bar{a}\in A_0\cup\{\bar{b}\}} H_{\bar{a}}$ in $\bigcap_{\bar{a}\in I} H_{\bar{a}}$ is less than d_ϕ.*

Now, we analyze uniformly definable subgroups of another generalization of stable theories, namely dependent theories. Note again, that any theory which is dependent as well as simple is indeed stable and recall that $\phi(x, \bar{b})$ defines a subgroup of G for every choice of $\bar{b}$ which we denote by $H_{\bar{b}}$. In the case of dependent theories one obtains a slightly a weaker chain condition known as the Baldwin-Saxl Condition. Moreover, if any uniformly definable family of subgroups of G satisfies the ICC^0 and the Baldwin-Saxl condition, one can derive the full ICC (Fact 2.5).

Fact 2.4 (Baldwin-Saxl Condition [3])**.** *If T is dependent then there is some natural number n_ϕ such that for any finite parameter set $A \subset M^{|\bar{y}|}$ there exists $A_0 \subseteq A$ of size less or equal to n_ϕ such that*

$$\bigcap_{\bar{a}\in A_0} H_{\bar{a}} = \bigcap_{\bar{a}\in A} H_{\bar{a}}$$

Proof. Assume that the statement is false and let n be an arbitrary natural number. Then there exists a parameter set A of size at least n such that for every $\bar{b} \in A$

$$\bigcap_{\bar{a}\in A\setminus\{\bar{b}\}} H_{\bar{a}} \neq \bigcap_{\bar{a}\in A} H_{\bar{a}}$$

Hence, we can choose $g_{\bar{b}} \in \bigcap_{\bar{a}\in A\setminus\{\bar{b}\}} H_{\bar{a}} \setminus H_{\bar{b}}$ for every $\bar{b} \in A$. For $B \subset A$, we define g_B to be $\prod_{\bar{b}\in B} g_{\bar{b}}$. As $g_{\bar{b}} \in H_{\bar{a}}$ if and only if $\bar{a} \neq \bar{b}$, it is easy to see that $g_B \in H_{\bar{a}}$ if and only if $\bar{a} \notin B$. Thus, we can conclude by compactness that the formula $\phi(x,\bar{y})$ is not dependent. □

Note that we index n_ϕ by ϕ to emphasize that n_ϕ only depends on the formula ϕ.

Fact 2.5. *Suppose that any family of uniformly definable subgroups of G satisfies the Baldwin-Saxl condition and the ICC^0. Then they satisfy the ICC.*

Proof. Let $\mathfrak{H} = \{H_{\bar{a}} = \phi(G;\bar{a}) : \bar{a} \in A\}$ be a family of uniformly definable subgroups of G. Consider the following collection of definable subgroups of G:

$$\left\{\bigcap_{\bar{a}\in A_0} \phi(G;\bar{a}) : A_0 \text{ is a finite subset of } A\right\}$$

By the Baldwin-Saxl condition, each of these intersection is equal to a sub-intersection of size at most n for some fixed natural number n. Hence this set forms a uniformly definable family of subgroups of G and satisfies the ICC^0. So we can find a finite subset A_0 of A and a natural number d such that for any finite subset A_1 of A we have that

$$\left[\bigcap_{\bar{a}\in A_0} \phi(G;\bar{a}) : \bigcap_{\bar{a}\in A_0\cup A_1} \phi(G;\bar{a})\right] \leq d$$

Therefore we can find a finite subset A_2 of A such that this index is maximal and thus any bigger intersection of subgroups in $\mathfrak{H}$ has to be equal to $\bigcap_{\bar{a}\in A_2} \phi(G;\bar{a})$. Hence

$$\bigcap_{\bar{a}\in A_2} \phi(G;\bar{a}) = \bigcap_{\bar{a}\in A} \phi(G;\bar{a})$$

and we obtain the ICC for $\mathfrak{H}$. □

As in the case of stable theories, Shelah used the Baldwin-Saxl condition to show the existence of definable envelopes of abelian subgroups in some sufficiently saturated extension [59]. Secondly, this chain condition was one of the key tools in the proof of Kaplan, Scanlon and Wagner showing that an infinite field of positive characteristic which is definable in a dependent theory is Artin-Schreier closed [34].

In Chapter 7 of this thesis, we want to study groups and fields in n-dependent theories (Definition 1.8). One of the key tools is to find a suitable version of the Baldwin-Saxl condition for n-dependent formulas which can be found below. In this case, we have to work with a different family of uniformly definable subgroups as the formula has to be of a different shape.

Proposition 2.6. *Suppose that $\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ is an n-dependent formula for which the set $\psi(\bar{b}_0, \ldots, \bar{b}_{n-1}; G)$ defines a subgroup of G for any parameters $\bar{b}_0, \ldots, \bar{b}_{n-1}$. Then there exists a natural number m_ψ such that for any $d \geq m_\psi$ and any array $(\bar{a}_{i,j} : i < n,\ j \leq d)$ of tuples with $\bar{a}_{i,j}$ of length $|\bar{b}_i|$, there is $\nu \in d^n$ such that*

$$\bigcap_{\eta \in d^n} H_\eta = \bigcap_{\eta \in d^n, \eta \neq \nu} H_\eta$$

where $H_\eta = \psi(\bar{a}_{0,i_0}, \ldots, \bar{a}_{n-1,i_{n-1}}; G)$ for $\eta = (i_0, \ldots, i_{n-1})$.

Proof. Suppose, towards a contradiction, that for an arbitrarily large natural number m one can find a finite array $(\bar{a}_{i,j} : i < n,\ j \leq m)$ of parameters such that $\bigcap_{\eta \in m^n} H_\eta$ is strictly contained in any of its proper subintersections. Hence, for every $\nu \in m^n$ there exists c_ν in $\bigcap_{\eta \neq \nu} H_\eta \setminus \bigcap_\eta H_\eta$.

Now, for any subset J of m^n, we let $c_J := \prod_{\eta \in J} c_\eta$. Note that $c_J \in H_\nu$ whenever $\nu \in m^n \setminus J$. On the other hand, if ν is an element of J, all factors of the product except for c_ν belong to H_ν, whence $c_J \notin H_\nu$. By compactness, one can find an infinite array of parameters $(\bar{a}_{i,j} : i < n,\ j \leq \omega)$ and elements $\{c_J : J \subset \omega^n\}$ such that c_J belongs to H_ν if and only if $\nu \notin J$. Hence, the formula $\neg\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ has the n-independence property and whence the original formula $\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ has the n-independence property as well contradicting the assumption. □

As we have pointed out in the introduction, the class of NTP$_2$ theories contains all dependent as well as simple theories and seems to be the right generalization of the two. As a chain condition, one expects that a certain amalgamation of the two chain conditions which were found in dependent theories and simple theories will hold for NTP$_2$ theories. In fact, Chernikov, Kaplan and Simon proved the following:

Fact 2.7 (Chernikov, Kaplan, Simon [12]). *If T is an NTP$_2$ theory and for any $\bar{b}$ the set $H_{\bar{b}}$ defines a **normal** subgroup of G, then there are some natural numbers n_ϕ and d_ϕ such that for all finite sets $A \subset M^{|\bar{y}|}$ of size bigger than n_ϕ there exists $\bar{b} \in A$ such that*

$$\left[\bigcap_{\bar{a} \in A \setminus \{\bar{b}\}} H_{\bar{a}} : \bigcap_{\bar{a} \in A} H_{\bar{a}} \right] < d_\phi$$

Proof. Assume it is false. Thus, for all natural numbers n and d, we find a parameter set $A = \{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ of size n such that for all $i < n$, the index $[\bigcap_{\bar{a} \in A \setminus \{\bar{a}_i\}} H_{\bar{a}} : \bigcap_{\bar{a} \in A} H_{\bar{a}}] > d$. Set $H = \bigcap_{\bar{a} \in A} H_{\bar{a}}$ and $H_i = \bigcap_{j \neq i} H_{\bar{a}_j}$. Then for each $i \in n$, we can find $h_{ij} \in H_i$ for $j < d$ such that $\{h_{ij} \cdot H : j < d\}$ are pairwise distinct cosets of H in H_i.

- Hence, for $j \neq k$, we have $h_{ij} H_{\bar{a}_i} \cap h_{ik} H_{\bar{a}_i} = \emptyset$ as otherwise, $h_{ij}^{-1} h_{ik} \in H_{\bar{a}_i}$, whence $h_{ij}^{-1} h_{ik} \in H$ which leads to a contradiction by the choice of the h_{ij}'s.
- We claim that for every $f : n \to n$, the intersection $\bigcap_{i \in n} h_{if(i)} H_{\bar{a}_i}$ is non-empty. Multiplying by $h_{0f(0)}^{-1} \cdot \ \ldots \ \cdot h_{nf(n)}^{-1}$ on the right and using that each $H_{\bar{a}_i}$ is a normal subgroup of G, this is equivalent to $\bigcap_{i \in n} H_{\bar{a}_i}$ being non-empty which trivially holds.

Now, we let $\theta(x; \bar{y}, z) := \exists w(\phi(w, \bar{y}) \wedge x = z \cdot w)$, so $\theta(x; \bar{a}_i, h_{i,j})$ defines the coset $h_{i,j} H_{\bar{a}_i}$. Compactness yields that θ has TP_2 and we can conclude. $\square$

Corollary 2.8 (Chernikov, Kaplan, Simon [12]). *If T is an NTP_2 theory and for any $\bar{b}$ the set $\phi(G; \bar{b})$ defines a **normal** subgroup of G, then there is a natural number n_ϕ such that for any finite parameter set $A \subset M^{|\bar{y}|}$ there exists some $A_0 \subset A$ of size less or equal to n_ϕ such that*

$$\left[\bigcap_{\bar{a} \in A_0} H_{\bar{a}} : \bigcap_{\bar{a} \in A} H_{\bar{a}} \right] < \omega.$$

Proof. Let n_ϕ be the natural number given by Fact 2.7. If $|A| < n_\phi$, take A itself. If not, there exists $\bar{b}_0 \in A$ such that

$$\left[\bigcap_{\bar{a} \in A \setminus \{\bar{b}_0\}} H_{\bar{a}} : \bigcap_{\bar{a} \in A} H_{\bar{a}} \right] < \omega.$$

Iterating this process, we can find $\bar{b}_i \in A \setminus \{\bar{b}_j : j < i\}$ for $i < |A| - n_\phi$ such that

$$\left[\bigcap_{\bar{a} \in A \setminus \{\bar{b}_j : j \leq i\}} H_{\bar{a}} : \bigcap_{\bar{a} \in A \setminus \{\bar{b}_j : j < i\}} H_{\bar{a}} \right] < \omega.$$

Setting $A_0 := A \setminus \{\bar{b}_i : i < |A| - n_\phi\}$, we can conclude. $\square$

We can strengthen their chain condition, namely one can find n_ϕ and d_ϕ as above such that for any finite parameter set A one can not only find one element of A such that the index is bounded but find a subset A_0 of size n_ϕ such that for all $\bar{b} \in A$,

$$\left[\bigcap_{\bar{a} \in A_0} H_{\bar{a}} : \bigcap_{\bar{a} \in A_0 \cup \{\bar{b}\}} H_{\bar{a}} \right] < d_\phi.$$

Observe that for uniformly definable normal subgroups, this is exactly the combination of the chain conditions which hold in dependent and simple theories.

Proposition 2.9. *If T is an NTP_2 theory and for any $\bar{b}$ the set $\phi(G;\bar{b})$ defines a* ***normal*** *subgroup of G, then there exists n_ϕ, $d_\phi \in \omega$ such that for every finite parameter set $A \subset M^{|\bar{y}|}$ there exists $A_0 \subseteq A$ with $|A_0| \leq n_\phi$ such that for all $\bar{b} \in A$*

$$\left[\bigcap_{\bar{a}\in A_0} H_{\bar{a}} : \bigcap_{\bar{a}\in A_0\cup\{\bar{b}\}} H_{\bar{a}}\right] < d_\phi.$$

Proof. So let us assume that the contrary holds. Thus, suppose that for all n, $d \in \omega$ there exists a finite set A such that for all $A_0 \subseteq A$ with $|A_0| \leq n$ there is $\bar{b} \in A$ such that

$$\left[\bigcap_{\bar{a}\in A_0} H_{\bar{a}} : \bigcap_{\bar{a}\in A_0\cup\{\bar{b}\}} H_{\bar{a}}\right] \geq d.$$

Let n_ϕ be as in Corollary 2.8 and let n_0 and d be two arbitrary natural numbers greater than 0. We fix a finite set A which satisfies the above for $n = n_0 + n_\phi$ and d. Define:

$$\theta(x;\bar{y},z) := \exists u(\phi(u,\bar{y}) \wedge x = zu).$$

So for $\bar{a} \in A$ and $h \in G$, the formula $\theta(G;\bar{a},h)$ defines the coset $h \cdot H_{\bar{a}}$. The aim is to construct an $n_0 \times d$ array $(\bar{a}_{ij}, h_{ij})_{i\in d, j\in n_0}$ for which $\{\theta(x_i;\bar{a}_{ij},h_{ij})\}_{j\in d}$ is 2-inconsistent for all $i \in n_0$ and $\{\theta(x_i;\bar{a}_{ij_i},h_{ij_i})\}_{i\in n_0}$ is consistent for every choice of $(j_0,\ldots,j_{n_0-1}) \in d^{n_0}$. Thus, by compactness the formula $\theta(x;\bar{y},z)$ has TP_2 which contradicts the assumption.

By Corollary 2.8, we can find a subset A_0 of A of cardinality n_ϕ such that

$$\left[\bigcap_{\bar{a}\in A_0} H_{\bar{a}} : \bigcap_{\bar{a}\in A} H_{\bar{a}}\right] < \omega.$$

Choose $B' \subset A$ of cardinality $n-1$ containing A_0 such that the index $[\bigcap_{\bar{a}\in A_0} H_{\bar{a}} : \bigcap_{\bar{a}\in B'} H_{\bar{a}}]$ is maximal, let's say equal to m. By the choice of A, there is $\bar{x} \in A$ such that

$$\left[\bigcap_{\bar{a}\in B'} H_{\bar{a}} : \bigcap_{\bar{a}\in B'\cup\{\bar{x}\}} H_{\bar{a}}\right] \geq d.$$

Define B to be the union $B' \cup \{\bar{x}\}$. Let $\bar{b}$ be an arbitrary element which belongs to $B \setminus A_0$. As the index of $\bigcap_{\bar{a}\in B'} H_{\bar{a}}$ in $\bigcap_{\bar{a}\in A_0} H_{\bar{a}}$ was chosen to be maximal, we obtain the following diagram.

$$\begin{tikzcd}
 & \bigcap\limits_{\bar{a}\in A_0} H_{\bar{a}} \arrow[dl, dash, "\leq m"'] \arrow[dr, dash, "m"] \arrow[dd, dash, "\geq md"] & \\
\bigcap\limits_{\bar{a}\in B\setminus\bar{b}} H_{\bar{a}} \arrow[dr, dash] & & \bigcap\limits_{\bar{a}\in B'} H_{\bar{a}} \arrow[dl, dash, "\geq d"] \\
 & \bigcap\limits_{\bar{a}\in B} H_{\bar{a}} &
\end{tikzcd}$$

It implies that for all $b \in B \setminus A_0$, the index $[\bigcap_{\bar{a}\in B\setminus\{\bar{b}\}} H_{\bar{a}} : \bigcap_{\bar{a}\in B} H_{\bar{a}}]$ is greater or equal to d.

Now, let $B \setminus A_0 = \{\bar{a}_0, \dots, \bar{a}_{n_0-1}\}$. For every $\bar{b} \in B \setminus A_0$ we can choose $\{h_0^{\bar{b}}, \dots, h_{d-1}^{\bar{b}}\}$ a set of representatives of the different cosets of $\bigcap_{\bar{a}\in B\setminus\{\bar{b}\}} H_{\bar{a}}$ in $\bigcap_{\bar{a}\in B} H_{\bar{a}}$. Then

- For all $\bar{b} \in B \setminus A_0$ and $k, j \in d$ with $k \neq j$, we have $h_j^{\bar{b}} H_{\bar{b}} \cap h_k^{\bar{b}} H_{\bar{b}} = \emptyset$;
- Let $j_0, \dots, j_{n_0-1} \in d$. Set $h = h_{j_0}^{\bar{a}_0} \cdot \dots \cdot h_{j_{n_0-1}}^{\bar{a}_{n_0-1}}$. As $h_{j_i}^{\bar{a}_i} \in H_{\bar{a}_k}$ for $k \neq i$ and $H_{\bar{a}_k}$ is normal in G, we have that $h \in h_{j_i}^{\bar{a}_i} H_{\bar{a}_i}$ for all $i \in n_0$.

Hence the array $(\bar{a}_{ij}, h_{ij})$ with $\bar{a}_{ij} = \bar{a}_i$ and $h_{ij} = h_j^{\bar{a}_i}$ for $i \in n_0$ and $j \in d$ is as desired. □

2.2 Centralizers

A related field of study are groups in which chain conditions hold merely for centralizers.

Definition 2.10. A group G is called an $\mathfrak{M}_c$-group if for any parameter set $(a_i : i \in \omega)$, the intersection $\bigcap_{i\in\omega} C_G(a_i)$ is equal to a finite subintersection.

This is equivalent to any proper descending chain of centralizers stabilizing after finitely many steps or the existence of a natural number n such that $\bigcap_{i=0}^{n} C_G(a_i)$ is contained in $C_G(a_j)$ for all natural numbers j. Using this chain condition on centralizers, one still obtains definable envelopes for abelian and nilpotent groups [1] as well as nilpotency of the Fitting subgroup [15].

Similarly to $\mathfrak{M}_c$-groups, we define groups satisfying the ICC^0 merely for centralizers. One crucial difference to $\mathfrak{M}_c$-groups is that we demand that the ICC^0 passes onto definable sections and saturated extensions.

Definition 2.11. A group G is called $\widetilde{\mathfrak{M}}_c$*-group* if for any two definable subgroups H and N, such that N is normalized by H, there exists natural numbers n_{HN} and d_{HN} such that any chain of centralizers

$$C_{H/N}(h_0N) \geq \ldots \geq C_{H/N}(h_0N, \ldots, h_mN) \geq \ldots \quad (h_i \in H)$$

each having index at least d_{HN} in its predecessor has length at most n_{HN}.

To investigate the properties of $\widetilde{\mathfrak{M}}_c$-groups forms a big part of this thesis.

Remark 2.12. Note that any definable subgroup, any definable quotient and any elementary extension of and $\widetilde{\mathfrak{M}}_c$-group is again an $\widetilde{\mathfrak{M}}_c$-group.

3 Almost centralizer

In the next chapters, we want to study $\widetilde{\mathfrak{M}}_c$-groups. Examples are definable groups in simple theories, such as the theory of perfect bounded PAC-fields, (group theoretically) simple pseudofinite groups [64], or the extra special p-group (Example 4.3), as well as groups definable in rosy theories.

A useful notion in this context is the following: For a subgroup H of a group G, the *FC-centralizer* of H in G contains all elements of G whose centralizer in H has finite index in H. It was introduced by Haimo in [20]. We define a suitable version of this object for A-invariant subgroups of G which we call *almost centralizer*, and establish their basic properties.

Moreover, some of the results turn out to be key tools in finding definable envelopes for nilpotent subgroups of groups definable in an NTP_2 theory presented in Section 4.2.

3.1 Preliminaries

Let us first give the original definition of an FC-centralizer and related objects given by Haimo.

Definition 3.1. Let G be a group and H, K and N be three subgroups of G such that N is normalized by H. We define:

- The *FC-centralizer* of H in K modulo N:
$$\mathrm{FC}_K(H/N) = \{k \in N_K(N) : [H : C_H(k/N)] \text{ is finite}\}$$
- Suppose that $N \leq H \leq K$. Then, the n^{th} *FC-centralizer* of H in K modulo N is defined inductively on n as the following:
$$\begin{aligned} \mathrm{FC}^0_K(H/N) &= N \\ \mathrm{FC}^{n+1}_K(H/N) &= \mathrm{FC}_H(H/\mathrm{FC}^n_K(H/N)) \cap \bigcap_{i=0}^{n} N_K(\mathrm{FC}^i_K(H/N)) \end{aligned}$$
- The n^{th}*FC-center* of H:
$$\mathrm{FC}_n(H) = \mathrm{FC}^n_H(H)$$

Remark 3.2. The abbreviation FC stand for *finite conjugation*. This is related to the fact that an element g of a group G is in the FC-centralizer of a subgroup H of G if and only if the set of conjugates of g by elements in H is finite.

Definition 3.3. Let H and K be two arbitrary subgroups of G. We say that H is *virtually contained* in K, denoted by $H \leq_v K$ if the index of $H \cap K$ in H is finite. We say that H and K are *commensurable*, denoted by $H =_v K$, if H is virtually contained in K and K is virtually contained in H.

We want to generalize these notions to suitable versions of these objects and relations regarding A-invariant subgroups of G. For two such groups $H \leq K$, we have two options regarding the index of H in K: it is either *bounded*, i. e. it does not grow bigger than a certain cardinal while enlarging the ambient model, or for any given cardinal κ we can find an ambient model such that the index is larger than κ. Then we say that the index is *unbounded*. Note that if the index is bounded it is indeed bounded by $\left(2^{|T(A)|}\right)^+$. This leads to the definition below.

Definition 3.4. Let H and K be two A-invariant subgroups of G. We say that H is *almost contained* in K, denoted by $H \lesssim K$, if the index of $H \cap K$ in H is bounded. We say that H and K are *commensurate*, denoted by $H \sim K$, if H is almost contained in K and K is almost contained in H.

Let H and K be two A-invariant subgroups. Observe that $H \lesssim K$ does not depend on the model we choose. Thus $H \lesssim K$ remains true in any elementary extension. Moreover, if H and K are definable, bounded can be replaced by finite and hence being virtually contained and being almost contained coincide. Observe that being almost contained is a transitive relation and being commensurate is an equivalence relation among A-invariant subgroups of G. Furthermore, we have the following property:

Lemma 3.5. *Let G be a group and let H, K, and L be three A-invariant subgroups of G such that H normalizes K. If $H \lesssim L$ and $K \lesssim L$ then $HK \lesssim L$.*

Proof. We assume that G is sufficiently saturated. By assumption, we have that the index of $L \cap H$ in H as well as the index of $L \cap K$ in K are bounded by some cardinal κ_H and κ_K respectively which are smaller than $(2^{|T(A)|})^+$. Take $I_H = \{h_i : i < \kappa_H\}$ and $I_K = \{k_i : i < \kappa_K\}$ representatives of the cosets of $L \cap H$ in H and of $L \cap K$ in K respectively. Then the set $I_H \cdot I_K$ has at most size $2^{|T(A)|}$ and as H normalizes K, it contains a set of representatives of the cosets of $L \cap (HK)$ in HK. Hence the index of $L \cap (HK)$ in HK is bounded in G and so $HK \lesssim L$. □

Definition 3.6. Let H, K and N be three A-invariant subgroups of G such that N is normalized by H. We define:

- The *almost centralizer* of H in K modulo N:

$$\widetilde{C}_K(H/N) = \{g \in N_K(N) : H \sim C_H(g/N)\}$$

- The *almost center* of H:

$$\widetilde{Z}(H) = \widetilde{C}_H(H)$$

To prove the different properties of the almost centralizer, we make use of the Erdős-Rado theorem. To state it, let us first introduce the following notation:

Notation. Let κ be a cardinal. Then we define inductively:

$$\exp_0(\kappa) = \kappa \quad \text{and} \quad \exp_{r+1}(\kappa) = 2^{\exp_r(\kappa)} \text{ for } r \geq 0.$$

Moreover for cardinal κ, λ, δ and θ, we write

$$\kappa \longrightarrow (\lambda)^{\theta}_{\delta}$$

if for any coloring of the subsets of cardinality θ of a set of cardinality κ, in δ many colors, there is a homogeneous set of cardinality λ (a set, all whose subsets of cardinality θ get the same color).

Fact 3.7 (Erdős-Rado)**.** *[33, Theorem 9.6] Let n be a natural number and κ be an infinite cardinal, then*

$$\exp_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_{\kappa}.$$

Properties 3.8. Let H, H', K, L and L' be A-invariant subgroups of G such that H and H' normalize L and L'.

1. $\widetilde{C}_K(H)$ and $\widetilde{Z}(H)$ are A-invariant subgroups.
2. $C_G(H) \leq \widetilde{C}_G(H)$ and $Z(G) \leq \widetilde{Z}(G)$.
3. If H is definable, bounded can be replaced by finite and the almost centralizer and FC-centralizer coincide.
4. $\widetilde{C}_{H'}(H/L) = \widetilde{C}_G(H/L) \cap H'$.
5. $\widetilde{C}_G(H)$ is fixed by all definable automorphisms of G (in the pure language of groups) which fix H. Thus it is normalized by the normalizer of H and in particular by H. Furthermore, $\widetilde{Z}(H)$ is a definably-characteristic subgroup of H (i.e. fixed by all definable automorphism which fix H).
6. If $H \lesssim H'$ as well as $L \lesssim L'$ and $N_G(L) \leq N_G(L')$, we have that

   $$\widetilde{C}_G(H'/L) \leq \widetilde{C}_G(H/L')$$

   In particular,

   - $\widetilde{C}_G(H') \leq \widetilde{C}_G(H)$
   - $\widetilde{C}_G(H/L) \leq \widetilde{C}_G(H/L')$
7. Moreover, if $H \sim H'$ as well as $L \sim L'$ and $N_G(L) = N_G(L')$, we have that

   $$\widetilde{C}_G(H'/L) = \widetilde{C}_G(H/L')$$

   In particular,

   - $\widetilde{C}_G(H') = \widetilde{C}_G(H)$
   - $\widetilde{C}_G(H/L) = \widetilde{C}_G(H/L')$

8. Suppose that H is the union of A-type-definable subgroups H_α with $\alpha \in \Omega$. Then

$$\widetilde{C}_G(H) = \bigcap_{\alpha\in\Omega} \widetilde{C}_G(H_\alpha).$$

9. If L is the intersection of A-definable subgroups L_α of G with $\alpha \in \Omega$, we have that

$$\widetilde{C}_G(H/L) \cap \bigcap_{\alpha\in\Omega} N_G(L_\alpha) = \bigcap_{\alpha\in\Omega} \widetilde{C}_G(H/L_\alpha)$$

10. If L is the intersection of A-definable subgroups L_α of G with $\alpha \in \Omega$ all normalized by K and H,

$$H \lesssim \widetilde{C}_G(K/L) \quad \text{if and only if} \quad H \lesssim \widetilde{C}_G(K/L_\alpha) \quad \text{for all } \alpha \in \Omega.$$

Proof. 1. till 7. are obvious.

8. Observe first, that there are at most $2^{|T(A)|}$ many types over the fixed parameter set A and so the set Ω is bounded. Thus, if the centralizer of some element g in G has unbounded index in H by Erdős-Rado (Fact 3.7) there exists also an α in Ω such that $C_{H_\alpha}(g)$ has unbounded index in H_α. Hence g does not belong to $\widetilde{C}_G(H_\alpha)$. The converse is obvious.

9. The inclusion from left to right holds trivially. Now suppose that g is an element of $\bigcap_{\alpha\in\Omega} \widetilde{C}_G(H/L_\alpha)$. Then g belongs to $N_G(L_\alpha)$ by definition of the almost centralizer and g^H intersects only boundedly many cosets of L_α in H for all α in Ω. As the map $xL \mapsto (xL_\alpha : \alpha \in \Omega)$ is injective, the conjugacy class g^H of g intersects only boundedly many cosets of L and thus $g \in \widetilde{C}_G(H/L)$.

10. is an immediate consequence of (9). □

As for any normal subgroup N of H, we have that $\widetilde{C}_G(H/N)$ is normalized by H, the following definition of the iterated almost centralizers is well defined.

Definition 3.9. Let H and K be two A-invariant subgroups of G such that $H \leq K$ and N be a normal A-invariant subgroup of H, then

- The n^{th} *almost centralizer* of H in K modulo N is defined inductively on n by:

$$\begin{aligned}
\widetilde{C}^0_K(H/N) &= N \\
\widetilde{C}^{n+1}_K(H/N) &= \widetilde{C}_K(H/\widetilde{C}^n_K(H/N)) \cap \bigcap_{i=0}^{n} N_K(\widetilde{C}^i_K(H/N))
\end{aligned}$$

- The n^{th} *almost center* of H is defined as $\widetilde{Z}_n(H) = \widetilde{C}^n_H(H)$.

Note that if H and N are normal subgroups of K, the definition of the n^{th} almost centralizer of H in K modulo N simplifies to:

$$\widetilde{C}^0_K(H/N) = N \text{ and } \widetilde{C}^{n+1}_K(H/N) = \widetilde{C}_K(H/\widetilde{C}^n_K(H/N))$$

Properties 3.10. Let G be a group, $H \trianglelefteq K$ be two A-invariant subgroups of G and let $n \in \omega$. Then we have that

$$\widetilde{C}_K^n(H) = \widetilde{C}_G^n(H) \cap K.$$

Proof. We prove this by induction on n. For n equal to 1, this is Properties 3.8 (4). So suppose that $\widetilde{C}_K^n(H) = \widetilde{C}_G^n(H) \cap K$. Now we have that

$$\begin{array}{rcl} \widetilde{C}_K^{n+1}(H) & = & \widetilde{C}_K(H/\widetilde{C}_K^n(H)) \\ & \overset{\text{ind}}{\underset{\text{hyp}}{=}} & \widetilde{C}_K(H/\widetilde{C}_G^n(H) \cap K) \\ & \underset{H \trianglelefteq K}{=} & \left\{k \in K : H \sim C_H\left(k/(\widetilde{C}_G^n(H) \cap K)\right)\right\} \\ & \underset{3.8(4)}{=} & \left\{k \in N_G\left(\widetilde{C}_G^n(H)\right) : H \sim C_H\left(k/\widetilde{C}_G^n(H)\right)\right\} \cap K \\ & = & \widetilde{C}_G(H/\widetilde{C}_G^n(H)) \cap K \\ & = & \widetilde{C}_G^{n+1}(H) \cap K. \end{array}$$

□

In the rest of the section, we show properties of the almost centralizer of *ind-definable* subgroups of G. It is a model theoretic notion which generalizes type-definable subgroups and which falls into the class of invariant subgroups.

Definition 3.11. Let G be a group and A be a parameter set. An *A-ind-definable* subgroup H of G is the union of a directed system of A-type-definable subgroups of G, i. e. there is a family $\{H_\alpha : \alpha \in \Omega\}$ of A-type-definable subgroups of G such that for all α and β in Ω there is γ in Ω such that $H_\alpha \cup H_\beta \leq H_\gamma$ and H is equal to $\bigcup_{\alpha \in \Omega} H_\alpha$.

3.2 Symmetry

Observe that for two subgroups H and K of a group G, we have trivially that $H \leq C_G(K)$ if and only if $K \leq C_G(H)$. In the case of FC-centralizers and virtual containment, we will see that this is not true for arbitrary subgroups in non-saturated models. However, we obtain the same symmetry condition replacing the centralizer by the almost centralizer and containment by almost containment for ind-definable subgroups. In case, the ambient theory is simple, this was proven by Palacín and Wagner in [50].

We use the following fact due to B. Neumann.

Fact 3.12. *[49, Lemma 4.1] A group cannot be covered by finitely many cosets of subgroups of infinite index.*

Theorem 3.13 (Symmetry)**.** *Let G be a group, H and K be two A-ind-definable subgroups of G and let N be a subgroup of G which is a union of A-definable sets. Suppose N is normalized by H and by K. Then*

$$H \lesssim \widetilde{C}_G(K/N) \quad \textit{if and only if} \quad K \lesssim \widetilde{C}_G(H/N).$$

Proof. Let κ be equal to $2^{|T(A)|}$ and assume that G is $(2^\kappa)^+$-saturated. We suppose that K is not almost contained in $\widetilde{C}_G(H/N)$. We want to show that H is not almost contained in $\widetilde{C}_G(K/N)$. By assumption, there is a set of representatives $\{k_i : i \in (2^\kappa)^+\}$ in K of different cosets of $\widetilde{C}_K(H/N)$ in K as G is sufficiently saturated. Note that H is the union of type-definable subgroups H_α with α in an index set Ω of cardinality at most κ. Thus for every i different than j in $(2^\kappa)^+$, there is $\alpha_{(i,j)}$ in Ω such that the centralizer of the element $k_i^{-1}k_j/N$ has unbounded index in $H_{\alpha_{(i,j)}}$. By Erdős-Rado (Fact 3.7), we can find a subset I_0 of $(2^\kappa)^+$ of cardinality κ^+ and α in Ω such that for all distinct i and j in I_0, we have that $\alpha_{(i,j)}$ is equal to α and thus the centralizer $C_{H_\alpha}(k_i^{-1}k_j/N)$ has infinite index in H_α. Hence, H_α can not be covered by finitely many cosets of these centralizers by Fact 3.12. As additionally the complement of N is type-definable the following partial type is consistent:

$$\pi(x_n : n \in \kappa^+) = \left\{[x_n^{-1}x_m, k_i^{-1}k_j] \notin N : n \neq m \in \kappa^+, i \neq j \in I_0\right\} \cup \{x_n \in H_\alpha : n \in \kappa^+\}$$

As G is sufficiently saturated, one can find a tuple $\bar{h}$ in G which satisfies $\pi(\bar{x})$. Fix two different elements n and m in κ^+. Then, we have that $k_i^{-1}k_j \notin C_K(h_n^{-1}h_m/N)$ for all $i \neq j$ in I_0. Hence, the subgroup $C_K(h_n^{-1}h_m/N)$ has unbounded index in K witnessed by $(k_j : j \in I_0)$, and whence the element $h_n^{-1}h_m$ does not belong to $\widetilde{C}_H(K/N)$. So $\widetilde{C}_H(K/N)$ has unboudedly many H_α-translates and therefore unbounded index in H. Thus, the group H is not almost contained in $\widetilde{C}_G(K/N)$ which finishes the proof. □

We obtain the following useful corollary.

Corollary 3.14. *Let G be an $\aleph_0$-saturated group and H and K be two definable subgroups of G. Then*

$$H \leq_v \widetilde{C}_H(K) \text{ if and only if } K \leq_v \widetilde{C}_K(H)$$

Proof. Since almost containment and the almost centralizer satisfies symmetry, it is enough to show that for definable subgroups H and K of an $\aleph_0$-saturated group, we have that

$$H \leq_v \widetilde{C}_H(K) \text{ if and only if } H \lesssim \widetilde{C}_H(K).$$

So suppose first that $H \leq_v \widetilde{C}_H(K)$ and fix representatives $h_1, \ldots, h_n$ of the distinct classes of $\widetilde{C}_H(K)$ in H. Let H_d be the definable set $\{h \in H : [K : C_K(h)] < d\}$. As K is definable, we have that $\widetilde{C}_H(K) = \bigcup_{d\in\omega} H_d$. Thus

$$H = \bigcup_{i=1}^{n} h_i \cdot \bigcup_{d\in\omega} H_d.$$

By $\aleph_0$-saturation, this remains true in any elementary extension of G and so $H \lesssim \widetilde{C}_H(K)$.

On the other hand, if $H \not\leq_v \widetilde{C}_H(K)$, then for any cardinal κ the type

$$\pi(x_i : i \in \kappa) = \{x_i \in H\} \cup \{x_i^{-1}x_j \notin H_d : i \neq j, d \in \omega\}$$

is consistent. Hence, $H \not\lesssim \widetilde{C}_H(K)$. □

In the general context, we may ask if symmetry holds for FC-centralizers. We will give a positive answer in the case that the ambient group is an $\mathfrak{M}_c$-group. Afterwards, we give a counter-example which shows that it does not hold in general.

Proposition 3.15. *Let G be an $\mathfrak{M}_c$-group and H and K be subgroups of G. Then*

$$H \leq_v \mathrm{FC}_G(K) \quad \textit{if and only if} \quad K \leq_v \mathrm{FC}_G(H).$$

Proof. Suppose that $H \leq_v \mathrm{FC}_G(K)$. So the group $\mathrm{FC}_H(K)$ has finite index in H and is obviously contained in $\mathrm{FC}_G(K)$. Note that by the former the FC-centralizer of $\mathrm{FC}_H(K)$ in K is equal to the one of H in K. Since G is an $\mathfrak{M}_c$-group, we can find elements $h_0, \ldots, h_n$ in $\mathrm{FC}_H(K)$ such that $C_G(\mathrm{FC}_H(K))$ is equal to the intersection of the centralizers of the h_i's. As each h_i is contained in the FC-centralizer of K in H, this intersection and hence $C_K(\mathrm{FC}_H(K))$ has finite index in K. In other words, K is virtually contained in $C_K(\mathrm{FC}_H(K))$ which, on the other hand, is trivially contained in $\mathrm{FC}_K(\mathrm{FC}_H(K))$. As $\mathrm{FC}_K(\mathrm{FC}_H(K))$ coincides with $\mathrm{FC}_K(H)$ as mentioned before we can conclude. □

The next example was suggested by F. Wagner.

Example 1. Let G be a finite non-commutative group, K be $\prod_\omega G$ and H be the subgroup $\bigoplus_\omega G$ of K. The support of an element $(k_i)_{i\in\omega}$ in K, denoted by $\mathrm{supp}((k_i)_{i\in\omega})$, is the set of indices $i \in \omega$ such that k_i is non trivial. As any element $\bar{h}$ of H has finite support and G is finite, any element of H has finitely many conjugates in K, namely at most $|G|^{|\mathrm{supp}(\bar{h})|}$ many. Thus its centralizer has finite index in K. Hence H is contained in the FC-centralizer of K. On the other hand, fix an element g of G which is not contained in the center of G. Let $\bar{k}_0$ be the neutral element of K and for $n \geq 1$ we define:

$$\bar{k}_n = (k_i)_{i\in\omega} \quad \text{such that} \quad \begin{cases} k_i = g & \text{if } i \equiv 0 \pmod{n} \\ k_i = 1 & \text{else} \end{cases}$$

Now fix some distinct natural numbers n and m. We have that the element $\bar{k}_n^{-1}\bar{k}_m$ is a sequence of the neutral element of G and infinitely many g's or g^{-1}'s. Now, we can choose an element h in G which does not commute with g and for any j in the support of $\bar{k}_n^{-1}\bar{k}_m$ we define the following elements of H:

$$\bar{l}_j = (l_i)_{i\in\omega} \quad \text{such that} \quad \begin{cases} l_i = h & \text{if } i = j \\ l_i = 1 & \text{else} \end{cases}$$

These elements witness that the set of conjugates $(\bar{k}_n^{-1}\bar{k}_m)^H$ is infinite and, as the n and m were chosen arbitrary, the $\bar{k}_n$'s are representatives of different cosets of $FC_K(H)$ in K. Thus K is not virtually contained in the FC-centralizer of H in K which contradicts symmetry.

The previous example demonstrates that symmetry does not hold for the FC-centralizer of arbitrary subgroups in non-saturated models but the following question still remains open:

Question 1. Let H and K be two A-invariant subgroups of a group G. Then, do we have that

$$H \lesssim \widetilde{C}_G(K) \quad \text{if and only if} \quad K \lesssim \widetilde{C}_G(H)\,?$$

3.3 The almost three subgroups lemma

For subgroups H, K and L of some group G we have that

$$[H,K,L] = 1 \text{ and } [K,L,H] = 1 \text{ imply } [L,H,K] = 1,$$

which is known as the three subgroups lemma. We want to generalize this result to our framework. As we have not yet introduced an "almost" version of the commutator, observe that, if H, K, and L normalize each other, we have that $[H,K,L] = 1$ if and only if $H \leq C_G(K/C_G(L))$. Thus we may state the three subgroups lemma as follows:

$$H \leq C_G(K/C_G(L)) \text{ and } K \leq C_G(L/C_G(H)) \text{ imply } L \leq C_G(H/C_G(K)).$$

This statement, replacing all centralizers and containment by almost centralizers and almost containment, can be deduced from the lemma proven below in the case of ind-definable subgroups if they normalize each other in the following sense:

Definition 3.16. Let H, K and L be three A-ind-definable subgroups of G. We say that

- H *strongly normalizes* L if there is a set of A-type-definable subgroups $\{L_\alpha : \alpha \in \Omega\}$ of G each normalized by H such that L is equal to $\bigcup_{\alpha\in\Omega} L_\alpha$.
- H and K *simultaneously strongly normalize* L if there is a set of A-type-definable subgroups $\{L_\alpha : \alpha \in \Omega\}$ of G each normalized by H and K such that L is equal to $\bigcup_{\alpha\in\Omega} L_\alpha$.
- L is a *strongly normal* subgroup of G if G strongly normalizes L.

Note that if L is a type-definable group, it is strongly normalized by H (or respectively simultaneously strongly normalized by H and K) if and only if H normalizes L (respectively H and K normalize L).

Lemma 3.17. *Let H, K and L be three A-ind-definable subgroups of G. If H and K simultaneously strongly normalize L, then the following is equivalent:*

- $H \not\lesssim \widetilde{C}_G\big(K/\widetilde{C}_G(L)\big)$.
- *For any cardinal κ, there exists an elementary extension $\mathcal{G}$ of G and elements $(h_i : i \in \kappa)$ in $H(\mathcal{G})$, $(k_n : n \in \kappa)$ in $K(\mathcal{G})$ and $(l_s : s \in \kappa)$ in $L(\mathcal{G})$ such that*

$$[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t] \neq 1 \quad \forall i,j,n,m,s,t \in \kappa,\ i \neq j, n \neq m, s \neq t.$$

Proof. Let $\{L_\alpha : \alpha \in \Omega_L\}$ be a set of A-type-definable subgroups of G each normalized by H and K such that L is equal to $\bigcup_{\alpha\in\Omega_L} L_\alpha$ and $\{K_\beta : \beta \in \Omega_K\}$ be a set of A-type-definable subgroups of G such that K is equal to $\bigcup_{\beta\in\Omega_K} K_\beta$. Assume first that $H \not\lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$. Note that as K and H normalize L, they normalize as well $\widetilde{C}_G(L)$. So $\widetilde{C}_G(K/\widetilde{C}_G(L))$ is well defined and for any $h \notin \widetilde{C}_H(K/\widetilde{C}_G(L))$, we have that $[K : C_K(h/\widetilde{C}_G(L))]$ is unbounded by the definition of the almost centralizer.

Let κ be a given cardinal greater than $(2^{|T(A)|})^+$. Assume that G is $(2^{(2^\kappa)})^+$-saturated. The goal is to find elements $(h_i : i \in \kappa)$ in H, $(k_n : n \in \kappa)$ in K and $(l_s : s \in \kappa)$ in L which satisfy the second condition of the Lemma.

By saturation of G, one can find a sequence $(h_i : i \in (2^{(2^\kappa)})^+)$ of elements in H such that for non equal ordinals i and j, the element $h_i^{-1}h_j$ does not belong to $\widetilde{C}_G(K/\widetilde{C}_G(L))$ or equivalently

$$K \not\lesssim C_K(h_i^{-1}h_j/\widetilde{C}_G(L)). \quad (*)$$

Claim. *There is a subset I of $(2^{(2^\kappa)})^+$ of size κ^+, $\beta \in \Omega_K$ and $\alpha \in \Omega_L$ such that for all distinct elements i and j in I, we have that $K_\beta \not\lesssim C_{K_\beta}(h_i^{-1}h_j/\widetilde{C}_G(L_\alpha))$.*

Proof of the claim. Let i and j be two different arbitrary ordinal numbers less than $(2^{(2^\kappa)})^+$. By $(*)$ there exists a sequence $(k_n^{(i,j)} : n \in (2^\kappa)^+)$ of elements in K such that for non identical ordinals n and m less than $(2^\kappa)^+$, we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G(L).$$

As K is the bounded union of A-type-definable subgroups K_β, by the pigeon hole principle we can find subset $J_{i,j}$ of $(2^\kappa)^+$ of size $(2^\kappa)^+$ and $\beta_{i,j}$ in Ω_K such that for all n in $J_{i,j}$, the element $k_n^{(i,j)}$ is an element of $K_{\beta_{i,j}}$. To simplify notation we may assume that $J_{i,j}$ is equal to $(2^\kappa)^+$. Now, by Erdős-Rado (Fact 3.7), we can find a subset I of $(2^{(2^\kappa)})^+$ of size $(2^\kappa)^+$ and $\beta \in \Omega_K$ such that for i different from j in I, we have that $\beta_{i,j}$ is equal β. Again for convenience we assume that I equals $(2^\kappa)^+$.

To summarize, we have now found β in Ω_K, a sequence of elements $(h_i : i \in (2^\kappa)^+)$ in H and for any i different than j in $(2^\kappa)^+$ a sequence $(k_n^{(i,j)} : n \in (2^\kappa)^+)$ in K_β such that

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G(L).$$

Fix again two distinct ordinal numbers i and j in $(2^\kappa)^+$. By Properties 3.8 (8), we have that the almost centralizer of L in G is the intersection of the almost centralizers of the L_α's in G. So for any non equal n and m in $(2^\kappa)^+$ one can find $\alpha_{(n,m)}^{(i,j)}$ in Ω_L such that

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_{\alpha_{(n,m)}^{(i,j)}}\right).$$

Now, we apply Erdős-Rado (Fact 3.7) to the sequences of the $k_n^{(i,j)}$'s. Doing so, we obtain a subset $I_{(i,j)}$ of $(2^\kappa)^+$ of cardinality at least κ^+ and $\alpha_{(i,j)}$ in Ω_L such that for all non identical n and m in $I_{(i,j)}$, we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_{\alpha_{(i,j)}}\right).$$

Next, we apply Erdős-Rado (Fact 3.7) to the h_i's. So, there exists a subset I of $(2^\kappa)^+$ of cardinality at least κ^+ and α in Ω_L such that $\alpha_{(i,j)}$ is equal to α for i different than j in I and thus for any such tuples we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_\alpha\right).$$

Thus, as for all non equal i and j in I, the index set $I_{(i,j)}$ is of cardinality $\kappa^+ > (2^{|T(A)|})^+$, we conclude that the centralizer of the element $h_i^{-1}h_j/\widetilde{C}_G(L_\alpha)$ has infinite index in K_β (witnessed by the $k_n^{(i,j)}$'s). Hence, for all distinct i and j in the index set I of cardinality κ^+, we have that $K_\beta \not\lesssim C_{K_\beta}(h_i^{-1}h_j/\widetilde{C}_G(L_\alpha))$ and the claim is established. $\square_{\text{claim}}$

The claim together with Fact 3.12 yield that the group $K_\beta/\widetilde{C}_G(L_\alpha)$ can not be covered by finitely many translates of these centralizers.

Now, observe that since L_α is a type-definable group, any relatively definable subgroup of L_α has either finite or unbounded index, whence the group $\widetilde{C}_G(L_\alpha)$ is equal to the union of the following definable sets

$$S_{\phi,d} = \left\{ g \in G : \forall l_0, \dots, l_d \bigwedge_{i=0}^{d} \phi(l_i) \to \bigvee_{i \neq j} l_i^{-1} l_j \in C_G(g) \right\},$$

where $\phi(x)$ ranges over the formulas in the type $\pi_{L_\alpha}(x)$ which defines L_α and d over all natural numbers.

By the two previous paragraphs, we conclude that the partial type below is consistent.

$$\begin{aligned} \pi(x_n : n \in \kappa) &= \{[h_i^{-1}h_j, x_n^{-1}x_m] \notin S_{\phi,d} : n \neq m \in \kappa, i \neq j \in I, d \in \omega, \phi \in \pi_{L_\alpha}\} \\ &\quad \cup \{x_n \in K_\beta : n \in \kappa\} \end{aligned}$$

Take $\bar{k}$ which satisfies $\pi(\bar{x})$. By construction we have that $[h_i^{-1}h_j, k_n^{-1}k_m] \notin \widetilde{C}_G(L_\alpha)$. Hence, $L_\alpha \not\lesssim C_{L_\alpha}([h_i^{-1}h_j, k_n^{-1}k_m])$. So L_α cannot be covered by finitely many translates of these centralizers. So the partial type below is again consistent.

$$\begin{aligned} \pi'(x_s : s \in \kappa) &= \{[[h_i^{-1}h_j, k_i^{-1}k_j], x_s^{-1}x_t] \neq 1 : s \neq t \in \kappa, n \neq m \in \kappa, i \neq j \in I,\} \\ &\quad \cup \{x_s \in L_\alpha : s \in \kappa\} \end{aligned}$$

As L_α is a subgroup of L, a realization of this type together with the $(h_i : i \in I)$ and $(k_n : n \in \kappa)$ satisfies the required properties.

On the other hand, suppose that for any cardinal κ, there exists an extension $\mathcal{G}$ of G and elements $(h_i : i \in \kappa)$ in $H(\mathcal{G})$, $(k_n : n \in \kappa)$ in $K(\mathcal{G})$, and $(l_s : s \in \kappa)$ in $L(\mathcal{G})$ such that

$$[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t] \neq 1 \quad \forall i, j, n, m, s, t \in \kappa,\ i \neq j, n \neq m, s \neq t.$$

So let κ be greater than $2^{|T(A)|}$. If $H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$ then one can find $i \neq j$ such that $h_i^{-1}h_j$ is an element of $\widetilde{C}_G(K/\widetilde{C}_G(L))$. So the index of $C_K(h_i^{-1}h_j/\widetilde{C}_G(L))$ in K is bounded. Once more this implies that one can find $n \neq m$ such that $k_n^{-1}k_m \in C_G(h_i^{-1}h_j/\widetilde{C}_G(L))$. Thus $[h_i^{-1}h_j, k_n^{-1}k_m]$ is an element of $\widetilde{C}_G(L)$ or equivalently the index of $C_L([h_i^{-1}h_j, k_n^{-1}k_m])$ has bounded index in L. Thus there exists $s \neq t$ such that $[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t] = 1$ which contradicts our assumption and the Lemma is established. $\square$

Now we are ready to prove the almost three subgroups lemma. We use additionally Witt's identity:

Fact 3.18 (Witt's identity)**.** *[31, Satz 1.4] Let G be a group and x, y, z be elements of G. Then*

$$[x, y^{-1}, z]^y \cdot [y, z^{-1}, x]^z \cdot [z, x^{-1}, y]^x = 1.$$

In particular, if $[z,x^{-1},y]$ is non trivial then either $[x,y^{-1},z]$ or $[y,z^{-1},x]$ is non trivial as well.

Theorem 3.19 (almost three subgroup lemma)**.** *Let G be a group and H, K and L be three ind-definable subgroups of G which simultaneously strongly normalize each other. If*

$$H \lesssim \widetilde{C}_G\big(K/\widetilde{C}_G(L)\big) \text{ and } K \lesssim \widetilde{C}_G\big(L/\widetilde{C}_G(H)\big) \text{ then } L \lesssim \widetilde{C}_G\big(H/\widetilde{C}_G(K)\big).$$

Proof. Assume towards a contradiction that $L \not\lesssim \widetilde{C}_G(H/\widetilde{C}_G(K))$ and let κ be equal to $(2^{|T(A)|})^+$. By the previous lemma we can find $(l_s : s \in \exp_5(\kappa)^+)$ in L, $(k_n : n \in (\exp_5(\kappa)^+)$ in K and $(h_i : i \in \exp_5(\kappa)^+)$ in H in a sufficiently saturated extension of G such that

$$[[l_s^{-1}l_t, h_i^{-1}h_j], k_n^{-1}k_m] \neq 1 \quad \forall i,j,n,m,s,t \in (2^\kappa)^+,\ i \neq j, n \neq m, s \neq t.$$

By the Witt's identity (Fact 3.18), for every tuple $i<j<n<m<s<t<\exp_5(\kappa)^+$ either

$$[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] \neq 1 \quad \text{or} \quad [[k_n^{-1}k_m, l_t^{-1}l_s], h_j^{-1}h_i] \neq 1.$$

By Erdős-Rado (Fact 3.7) we can find a subset I of cardinality κ^+ such that for all $i<j<n<m<s<t$ in I the same inequality of the two holds, say $[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] \neq 1$. Now let λ be the order-type of I and note that it is greater or equal to κ^+. Identify I with λ. Thus

$$[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] \neq 1 \quad \text{for} \quad 0 \leq i<j \leq \kappa < n<m \leq 2\kappa < s<t \leq 3\kappa \tag{3.1}$$

Furthermore, by assumption we have that $H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$. Hence, we can find two ordinal numbers i and j with $i<j<\kappa$ and such that $h_j^{-1}h_i$ is an element of $\widetilde{C}_G(K/\widetilde{C}_G(L))$. So the index of $C_K(h_j^{-1}h_i/\widetilde{C}_G(L))$ in K is bounded. Once more this implies that there are two ordinal numbers n and m with $\kappa<n<m\leq 2\kappa$ and such that $k_m^{-1}k_n$ belongs to $C_G(h_j^{-1}h_i/\widetilde{C}_G(L))$. Thus $[h_j^{-1}h_i, k_m^{-1}k_n]$ is an element of $\widetilde{C}_G(L)$ or equivalently the index of $C_L([h_j^{-1}h_i, k_m^{-1}k_n])$ has bounded index in L. Thus there exists another two ordinal numbers s and t with $2\kappa<s<t\leq 3\kappa$ and such that $[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] = 1$. Finally, this contradicts (3.1) and the theorem is established. □

3.4 Generalized Neumann theorem

We want to generalize a classical group theoretical result due to B. H. Neumann. To do so, let us first introduce the following notions.

Definition 3.20. A group G is *finite-by-abelian* if there exists a normal finite subgroup F of G such that G/F is abelian.

Observe that G being finite-by-abelian is equivalent to its derived group $[G,G]$ being finite.

Definition 3.21. A group G is *almost abelian* if the centralizer of any of its elements has finite index in G. If there is a natural number d such that the index of the centralizer of any element of G in G is smaller than d, we say that G is a *bounded almost abelian group*.

Remark 3.22. If we consider a definable almost abelian subgroup of an $\aleph_0$-saturated group, we can always bound the index of the centralizers by some natural number d by compactness. Hence, any definable almost abelian subgroup of any $\aleph_0$-saturated group is a *bounded almost abelian group*. Additionally, note that the almost center of any group is always an almost abelian group.

Now, we can state the fact:

Fact 3.23. *[49, Theorem 3.1]. Let G be a bounded almost abelian group. Then its derived group is finite and thus G is finite-by-abelian.*

We realized that the main tool in the beginning of the proof is the existence of a natural number which bounds the size of the set of conjugates of any element of H in H. Secondly, it seems not to be of importance that one considers conjugates of H by itself, actually it works for two groups H and K such that H is contained in the almost centralizer of K and there is some natural number that bounds the number of conjugates of each element of K by H. This leads to the following theorem and corollary.

Theorem 3.24. *Let G be a group and let H and K be two subgroups of G. Suppose that*

- *H normalizes K;*
- *$H \leq \mathrm{FC}_G(K)$;*
- *$K \leq \mathrm{FC}_G(H)$, moreover there is $d \in \omega$ such that for all k in K the set of conjugates k^H has size at most d.*

Then the group $[K,H]$ is finite.

Remark 3.25. Let G be a bounded almost abelian group. So letting H and K be equal to G in the previous theorem, all assumption are met. Thus, we obtain that $[G,G]$ is finite and recover the theorem of Neumann.

In the proof, we use the following fact:

Fact 3.26. *[2, 53] Let G be a group and let K and H be two subgroups of G such that H normalizes K. If the set of commutators*

$$\{[k,h] : k \in K, h \in H\}$$

is finite, then the group $[K,H]$ is finite.

Proof of Theorem 3.24. Let d be the minimal bound for the size of conjugacy classes of elements of K by H. Fix some element k of K for which the conjugacy class of k in H has size d and let $1, h_2, \ldots, h_d$ be a set of right coset representatives of H modulo $C_H(k)$. Thus

$$k_1 = k, \;\; k_2 = k^{h_2}, \;\; \ldots, \;\; k_d = k^{h_d}$$

are the d distinct conjugates of k by H. We let C be equal to the centralizer $C_K(h_2, \ldots, h_d)$. As H is contained in $\mathrm{FC}_G(K)$, the group C has finite index in K. Choose some representatives $a_1, \ldots, a_n$ of right cosets of K modulo C. Note that their conjugacy classes by H

are finite by assumption. Let F be the finite set $k^H \cup a_1^H \cup \cdots \cup a_n^H$ and let E be the set $\{x_0 \cdot x_1 \cdot x_2 \cdot x_3 : x_i \in F \cup F^{-1}, i < 4\}$ which is finite as well. Note that K is equal to CF.

Now, we want to prove that E contains the set

$$D := \{[g,h] : g \in K, h \in H\}.$$

So let $g \in K$ and $h \in H$ be arbitrary elements. Choose c in C, f in F, such that $g = cf$. We have that

$$[g,h] = [cf,h] = [c,h]^f [f,h] = f^{-1}[c,h] \cdot f^h$$

As f^{-1} belongs to F^{-1} and f^h belong to F, it remains to show that $[c,h]$ can be written as a product of two elements in $F \cup F^{-1}$.

Let $w = ck$. As c commutes with $h_2, \ldots, h_d$ the conjugates

$$w = ck, \;\; w^{h_2} = ck_2, \;\; \ldots, \;\; w^{h_d} = ck_d$$

are all different. As d was chosen to be maximal, these have to be all conjugates of w by H. So there are i and j less or equal than d, such that

$$h^{-1}wh = ck_i \;\text{ and }\; h^{-1}kh = k_j.$$

We obtain that

$$[c,h] = c^{-1}h^{-1}ch = c^{-1}(h^{-1}ckh)(h^{-1}k^{-1}h) = c^{-1}ck_ik_j^{-1} = k_ik_j^{-1}.$$

As all k_i's belong to F, we can conclude that D is a subset of E and therefore finite. Hence $[K,H]$ is finite by Fact 3.26. □

Corollary 3.27. *Let G be an $\aleph_0$-saturated group and let H and K be two definable subgroups of G such that H normalizes K. Suppose that*

$$K \le \widetilde{C}_G(H) \;\; \textit{and} \;\; H \le \widetilde{C}_G(K).$$

Then the group $[K,H]$ is finite.

Proof. As G is $\aleph_0$-saturated, the fact that $K \le \widetilde{C}_G(H)$ implies that there is $d \in \omega$ such that for all k in K the set of conjugates k^H has size at most d. So all hypotheses of Theorem 3.24 are satisfied and we can conclude. □

3.5 $\widetilde{\mathfrak{M}}_c$-groups

One of the crucial properties of subgroups of an $\widetilde{\mathfrak{M}}_c$-group G is that the iterated almost centralizers are definable which we prove below.

Proposition 3.28. *Let G be an $\widetilde{\mathfrak{M}}_c$-group, let H be a subgroup of G and let N be a definable subgroup of G which is contained in and normalized by H.*

1. *Then all iterated FC-centralizers $FC_G^n(H/N)$ are definable.*

2. *If H is an A-invariant group, then all iterated almost centralizers $\widetilde{C}_G^n(H/N)$ are definable.*

Proof. The proofs for the two cases are identical just replacing the iterated almost centralizers by the iterated FC-centralizers and bounded by finite. We give the proof using the notion $\widetilde{C}_G^n(H/N)$.

For n equals to 0 there is nothing to show as N is definable by assumption.

Now, let $n \in \omega$ and assume that $\widetilde{C}_G^i(H/N)$ is definable for all $i \leq n$. This yields that $\bigcap_{i=0}^n N_G(\widetilde{C}_G^i(H/N))$ is a definable subgroup of G and thus an $\widetilde{\mathfrak{M}}_c$-group as well. Moreover, as $\widetilde{C}_G^{n+1}(H/N)$ only contains elements which belong to this intersection we may replace G by this intersection and assume that $\widetilde{C}_G^n(H/N)$ is a normal subgroup. Since G is an $\widetilde{\mathfrak{M}}_c$-group, there are $g_0, \ldots, g_m \in \widetilde{C}_G^{n+1}(H/N)$ and $d \in \omega$ such that for all $h \in \widetilde{C}_G^{n+1}(H/N)$:

$$\left[\bigcap_{i=0}^{i=m} C_G\Big(g_i/\widetilde{C}_G^n(H/N)\Big) : \bigcap_{i=0}^{i=m} C_G\Big(g_i/\widetilde{C}_G^n(H/N)\Big) \cap C_G\Big(h/\widetilde{C}_G^n(H/N)\Big)\right] < d$$

Let D be equal to the definable group $\bigcap_{i=0}^{i=m} C_G(g_i/\widetilde{C}_G^n(H/N))$ and let

$$S := \Big\{g \in G : \Big[D : C_D(g/\widetilde{C}_G^n(H/N))\Big] < d\Big\}$$

We show that the definable set S is equal to $\widetilde{C}_G^{n+1}(H/N)$. The inclusion $\widetilde{C}_G^{n+1}(H/N) \subset S$ is obvious by choice of the g_i's and d. So let $g \in S$. To prove the inverse inclusion, we may compute:

$$\begin{aligned}
[H : C_H(g/\widetilde{C}_G^n(H/N))] &\leq [H : H \cap D] \cdot \Big[H \cap D : C_{H\cap D}(g/\widetilde{C}_G^n(H/N))\Big] \\
&\leq [H : H \cap D] \cdot \Big[D : C_D(g/\widetilde{C}_G^n(H/N))\Big] \\
&< \infty \quad \text{(i. e. finite for 1. and bounded for 2.)}
\end{aligned}$$

Thus g belongs to $\widetilde{C}_G^{n+1}(H/N)$. Hence $\widetilde{C}_G^{n+1}(H/N)$ is equal to S, and whence definable. □

Remark 3.29. Note that all iterated almost centralizers of H in G are stabilized by any definable automorphism which fixes H set wise. So, if H is an A-invariant group, all its iterated almost centralizers are indeed definable over A. Moreover, for any (type-, ind-) definable (resp. A-invariant) subgroup H, the iterated almost centers of H are (type-, ind-) definable (resp. A-invariant).

4 Definable envelopes of subgroups

As pointed out in the introduction, finding definable sets around non-definable objects becomes very important, since it “brings” objects outside the scope of model theory into the category of definable sets.

In that sense, an ongoing line of research consists of finding “definable envelopes”. Specifically, one can ask if for a definable group G and a given abelian, nilpotent, or solvable subgroup of G, can one find a *definable* abelian, nilpotent, or solvable subgroup of G which contains the given subgroup. This is always possible in stable theories (see [52]), and one obtains slightly weaker results for simple and dependent theories.

In this chapter, we analyze arbitrary abelian, nilpotent and (normal) solvable subgroups of groups definable in NTP_2 theories and $\widetilde{\mathfrak{M}}_c$-groups. We prove the existence of definable envelopes up to finite index (if the ambient group is sufficiently saturated for NTP_2-theories), which is inspired by the result in simple theories (as well as the one in dependent theories).

4.1 Prelimaries

Let G be a group definable in a dependent theory, let H be a subgroup of G and suppose that G is $|H|^+$-saturated. The following two results summarize what we know about envelopes of H. The first was proven by Shelah in [59] and the second by de Aldama in [14].

Fact 4.1. *If H is abelian, then there exists a definable abelian subgroup of G which contains H.*

Fact 4.2. *If H is a nilpotent (respectively normal solvable) subgroup of G of class n, then there exists a definable nilpotent (respectively normal solvable) subgroup of G of class n which contains H.*

We now turn to the simple theory context. As the following remark shows, it is impossible to get envelopes in the same way one could achieve them in the stable and dependent case, and one must allow for some “finite noise”.

Remark 4.3. (see [46, 5.15-5.22]) Let T be the theory of an infinite vector space over $\mathbb{F}_p$ with $p > 2$ together with a non-degenerate skew symmetric bilinear form. Then T is supersimple of SU-rank 1 and in any model of T one can define an “extraspecial p-group” G, i. e. G is infinite, every non-trivial element of G has order p, the center of G is cyclic of order p and is equal to the derived group of G. This group has SU-rank 1 [46, Corollary 5.22]and as any centralizer has finite index, one can find an infinite abelian subgroup A. On the other hand, suppose that there is an abelian subgroup B of G which has finite index in G and let $g_0,\ldots,g_n$ be representatives of the different cosets of B in G. As the centralizer of any element of G has finite index in G, we conclude that $C_B(g_0,\ldots,g_n)$ virtually contains G. Hence $C_B(g_0,\ldots,g_n)$ is infinite and by the choice of B and $g_0,\ldots,g_n$, it has to be contained in the center which is finite by assumption. Thus there are no abelian subgroups of finite index in G. However, if G had a definable abelian subgroup B which contains A, that abelian group would have SU-rank 1, hence would be of finite index in G, a contradiction.

A model theoretic study of extra special p-groups can be found in [17].

So one has to modify the notion of definable envelopes which is adapted to the new context. In the abelian case, it is the following result proven by Milliet as [47, Proposition 5.6.].

Fact 4.4. *Let G be a group definable in a simple theory and let H be an abelian subgroup of G. Then there exists a definable finite-by-abelian subgroup of G which contains H.*

In the nilpotent and solvable case one must additionally take into account a "by finite" phenomenon which leads to the fact below also due to Milliet [46]:

Fact 4.5. *Let G be a group definable in a simple theory and let H be a nilpotent (respectively solvable) subgroup of G of class n. Then one can find a definable nilpotent (respectively solvable) subgroup of class at most $2n$ which virtually contains H.*

Note that if H is an abelian subgroup it is as well a nilpotent subgroup of class 1. So on the one hand, by Fact 4.4 we obtain that there is a definable finite-by-abelian group A which contains H. In fact the proof of Milliet gives even more: we have that $[A,A]$ is contained in the FC-center of A. Thus $C_A([A,A])$ is definable nilpotent group of class at most 2 which has finite index in A and thus virtually contains H. Thus in the case H is abelian, Fact 4.4 implies Fact 4.5.

By the following theorem due to Fitting, we obtain a stronger result for normal nilpotent subgroups:

Fact 4.6 (Fitting's Theorem). *[18] Let G be a group and H and K be two normal nilpotent subgroups of class n and m respectively. Then HK is a normal nilpotent subgroup of class at most $n+m$.*

So, assume that the nilpotent subgroup H of class n is additionally normal in the group G which has a simple theory. Then one can ask for the definable subgroup N, which virtually contains H, to be normal in G as well. Hence, the product of these two subgroups NH is a normal nilpotent subgroup of G of class at most $3n$ by Fitting's theorem and it obviously **contains** H. Moreover, it is definable as it is the finite union of translates of N by elements of H.

To find envelopes in the simple theory context, Milliet makes use of the definable version of a result proven by Schlichting in [55], which can be found in [63, Theorem 4.2.4]. It deals with families of uniformly commensurable subgroups.

Definition 4.7. A family $\mathcal{H}$ of subgroups is *uniformly commensurable* if there exists a natural number d such that for each pair of groups H and K from $\mathcal{H}$ the index of their intersection is smaller than d in both H and K.

Fact 4.8 (Schlichting's theorem). *Let G be a group and $\mathcal{H}$ be a family of definable uniformly commensurable subgroups. Then there exists a definable subgroup N of G which is commensurable with all elements of $\mathcal{H}$ and which is invariant under any automorphisms of G which stabilizes $\mathcal{H}$ setwise. Moreover, the group N is a finite extension of a finite intersection of elements in $\mathcal{H}$.*

We also make use of this fact both in the NTP_2 as well as the $\widetilde{\mathfrak{M}}_c$ context.

4.2 NTP_2 theories

The purpose of this section is to extend the results above to NTP_2 groups. This is joint work with Alf Onshuus.

Theorem 4.9. *Let G be a group definable in an NTP_2 theory, H be a subgroup of G and suppose that G is $|H|^+$-saturated. Then the following holds:*

1. *If H is abelian, then there exists a definable finite-by-abelian subgroup of G which contains H.*
   Furthermore, if H is normal in G, the definable finite-by-abelian subgroup can be chosen to be normal in G as well.

2. *If H is solvable of class n which is normal in G, then there exists a definable normal solvable subgroup S of G of class at most $2n$ which virtually contains H.*
   In particular, the group HS is a definable solvable subgroup of G of class at most $3n$ which contains H.

3. *If H is nilpotent of class n, then there exists a definable nilpotent subgroup N of G of class at most $2n$ which virtually contains H.*
   Moreover, if H is normal in G, the group N can be chosen to be normal in G as well and HN is a definable nilpotent group of class at most $3n$ which contains H.

Question 2. Is there an NTP_2 group G and an abelian, nilpotent or normal solvable subgroup H of size strictly larger than the saturation of G which does not admit a definable envelope in the sense of Theorem 4.9?
It there an NTP_2 group G and a solvable subgroup H such that G is $|H|^+$-saturated and H does not admit a definable envelope in the sense of Theorem 4.9?

In the abelian and solvable case we follow some of the ideas already present in the proof of de Aldama. Similarly to his proof and unlike the proof of Milliet in simple theories, we do not rely on a chain condition for intersections of uniformly definable subgroups, but we look to prove the result directly from the non existence of the array described in Definition 1.12. In the nilpotent case, we use additionally some properties of the almost centralizer which were shown in the previous chapter. They turn out to be the same tools needed to prove the corresponding result in $\widetilde{\mathfrak{M}}_c$-groups which is presented in the next section.

The following is the key lemma for the abelian case and it is used in the nilpotent case as well.

Lemma 4.10. *Let G be a group with an NTP$_2$ theory, let H be a subgroup of G and suppose that G is $|H|^+$-saturated. Let $\phi(x,y)$ be the formula $xy = yx$. Consider the following partial types:*

$$\pi_{Z(H)}(x) = \{\phi(x,g) : Z(H) \leq \phi(G,g),\ g \in G\}$$
$$\pi_H(x) = \{\phi(x,g) : H \leq \phi(G,g),\ g \in G\}.$$

Then there exists a natural number n such that

$$\pi_{Z(H)}(x_0) \cup \cdots \cup \pi_{Z(H)}(x_n) \cup \pi_H(y) \vdash \bigvee_{i \neq j} \phi(x_i^{-1}x_j, y).$$

Proof. Suppose that the lemma is false. Then for arbitrary large $n \in \omega$ one can find a sequence of tuples $(a_{\ell,0},\ldots,a_{\ell,n-1},b_\ell)_{\ell<\omega}$ in G such that for all $\ell \in \omega$

$$(\bar{a}_\ell, b_\ell) \models \pi_{Z(H)}(x_0) \cup \cdots \cup \pi_{Z(H)}(x_{n-1}) \cup \pi_H(y) \restriction \mathrm{dcl}(H \cup \{\bar{a}_k, b_k : k < \ell\})$$

and for all $0 \leq i < j < n$ we have that $a_{\ell,i}^{-1}a_{\ell,j} \notin C_G(b_\ell)$. We show that:

1. For all $i < n$ and all natural numbers $k \neq \ell$, we have that $a_{\ell,i} \in C_G(b_k)$;
2. For all $i,j < n$ and all $k < \ell < \omega$ we have that $a_{\ell,i} \in C_G(b_k^{a_{k,j}})$.

To do so, we let $k < \ell < \omega$ and $i,j < n$ be arbitrary and we prove that $a_{\ell,i} \in C_G(b_k)$ as well as $a_{k,i} \in C_G(b_\ell)$ and $a_{\ell,i} \in C_G(b_k^{a_{k,j}})$.

For any element z in $Z(H)$, we have that $H \leq C_G(z)$. Hence $\phi(x,z) \in \pi_H(x) \restriction H$. As b_k satisfies this partial type, we obtain that

$$Z(H) \leq C_G(b_k).$$

So $\phi(x,b_k)$ belongs to $\pi_{Z(H)}(x) \restriction \{b_k\}$. Since the element $a_{\ell,i}$ satisfies $\pi(x)_{Z(H)} \restriction H \cup \{b_k\}$, we get that $a_{\ell,i}$ belongs to $C_G(b_k)$.

On the other hand, if we take $a \in H$ we have that $Z(H)$ is a subgroup of $C_G(a)$ and thus $\phi(x,a) \in \pi_{Z(H)}(x) \restriction H$. As $a_{k,i}$ satisfies $\pi_{Z(H)}(x) \restriction H$, we obtain that

$$H \leq C_G(a_{k,i}).$$

So $\phi(x,a_{k,i}) \in \pi_H(x) \restriction \{a_{k,i}\}$. As the element b_ℓ satisfies $\pi(x)_H \restriction H \cup \{a_{k,i}\}$, we get that the element $a_{k,i}$ belongs to $C_G(b_\ell)$ which together with the previous paragraph yields (1).

As seen above, we have $Z(H) \leq C_G(b_k)$ and $H \leq C_G(a_{k,i})$. So $Z(H) \leq C_G(b_k^{a_{k,j}})$. Hence $\phi(x, b_k^{a_{k,j}})$ belongs to $\pi_{Z(H)}(x) \restriction \mathrm{dcl}(b_k, a_{k,j})$. Since $a_{\ell,i}$ satisfies $\pi(x)_{Z(H)} \restriction \mathrm{dcl}(b_k, a_{k,j})$, we obtain that $a_{\ell,i}$ belongs to $C_G(b_k^{a_{k,j}})$, which yields (2).

Let $\psi(x;y,z)$ be the formula that defines the coset $y{\cdot}C_G(z)$. We claim that the following holds:

- $\{\psi(x; a_{\ell,i}, b_\ell\} : i < n\}$ is 2-inconsistent for any $\ell \in \omega$;
- $\{\psi(x; a_{\ell,f(\ell)}, b_\ell\} : \ell \in \omega\}$ is consistent for any function $f : \omega \to n+1$.

The first family is 2-inconsistent as every formula defines a different coset of $C_G(b_\ell)$ in G. For the second we have to show that for all natural numbers m and all tuples $(i_0, \dots, i_m) \in n^m$ the intersection

$$a_{0,i_0} C_G(b_0) \cap \cdots \cap a_{m,i_m} C_G(b_m)$$

is nonempty. Using (1) and (2) and multiplying by $a_{0,i_0}^{-1} \cdot \cdots \cdot a_{m,i_m}^{-1}$ on the right, this is equivalent to $C_G(b_0^{a_{0,i_0}}) \cap \cdots \cap C_G(b_m^{a_{m,i_m}})$ being nonempty which is trivially true.

Compactness yields a contradiction to the fact that the group G has an NTP_2 theory and we obtain the result. □

Using compactness as well, we obtain the following immediate corollary.

Corollary 4.11. *Let G be a group with an NTP_2 theory, let H be a subgroup of G and suppose that G is $|H|^+$-saturated. Then there are finite tuples $\bar{a}$ and $\bar{b}$ in G and a natural number m such that*

- $Z(H) \leq C_G(\bar{a})$,
- $H \leq C_G(\bar{b})$,
- $\bigwedge_{i<m} x_i \in C_G(\bar{a}) \wedge y \in C_G(\bar{b}) \vdash \bigvee_{i \neq j} x_i^{-1} x_j \in C_G(y)$.

Abelian subgroups

Proof of Theorem 4.9(1). By Corollary 4.11, we can find finite tuples $\bar{a}$ and $\bar{b}$ in G and a natural number n such that $Z(H) \leq C_G(\bar{a})$, $H \leq C_G(\bar{b})$ and

$$\bigwedge_{i<m} x_i \in C_G(\bar{a}) \wedge y \in C_G(\bar{b}) \vdash \bigvee_{i \neq j} x_i^{-1} x_j \in C_G(y). \quad (*)$$

Since H is abelian, the definable subgroup $C_G(\bar{a}, \bar{b})$ of G contains H, and by $(*)$ this is a bounded almost abelian group. Thus, its commutator subgroup is finite by Fact 3.23, which yields the first assertion of Theorem 4.9(1). Moreover, if H is normal in G, the group $C_G(\bar{a}^G, \bar{b}^G)$ is a definable normal subgroup of G which still contains H and which is as well almost abelian. This completes the proof. □

Solvable subgroups

To prove the solvable case of Theorem 4.9 we introduce the following notations:

Definition 4.12. A group G is *almost solvable* if there exists a normal *almost abelian series* of finite length, i. e. a finite sequence

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

of normal subgroups of G such that G_{i+1}/G_i is an almost abelian group for all $i \in n$. The least such natural number $n \in \omega$ is called the *almost solvable class* of G.

Definition 4.13. Let G be a group and S be a definable almost solvable subgroup. We say that S *admits a definable almost abelian series of length* n if there exists a family of definable normal subgroups $\{S_i : i \leq n\}$ of S such that S_n is the trivial group, S_0 is equal to S and S_i/S_{i+1} is almost abelian and normalized by S.

In an arbitrary group, a priori not every almost solvable group admits a definable almost abelian series.

By the following Lemma we only need to concentrate on building a definable almost abelian series. The proof is analogous to the one of Corollary 4.12 in [46] (although it is done there in the context of a simple theory, the proof is exactly the same in our context).

Lemma 4.14. *Let G be an $\aleph_0$-saturated group and H be an almost solvable subgroup of G which admits a definable almost abelian series*

$$H = H_0 \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_n = \{1\}$$

of length n. Then H has a definable subgroup of finite index which is solvable of class at most $2n$ and which is normalized by $\bigcap_i N_G(H_i)$.

Proof. As the H_i are normalized by H, we may replace G by the definable $\bigcap_i N_G(H_i)$ and suppose that all H_i are normal in G. So, we need to find a definable finite index subgroup of H which is normal in G and solvable of class at most $2n$.

By compactness and saturation, we have that H_i/H_{i+1} are bounded almost abelian groups. Now, add the parameters needed to define the H_i to the language.

Using Fact 3.23 we deduce that the quotient group $[H_i, H_i]/H_{i+1}$ is finite. Moreover, as all H_i's are normal subgroups of G, the group $[H_i, H_i]/H_{i+1}$ is normalized by G. Hence, any h in $[H_i, H_i]$ has finitely many conjugates in H/H_{i+1}, i. e. the set $(h/H_{i+1})^H$ is finite. Thus the index of $C_H(h/H_{i+1})$ in H is finite. Hence, the definable group $C_H([H_i, H_i]/H_{i+1})$ is the finite intersection of centralizers which have finite index in H and whence it has finite index in H as well. Moreover, it is normalized by G as H, H_i and H_{i+1} are normal subgroups of G. We conclude that it contains the intersection of all definable G-normalized subgroups of H which have finite index in H which we denote by H^0. This implies that

$$[[H_i, H_i], H^0] \leq H_{i+1}.$$

Now, we show by induction on k that

$$(H^0)^{(2k)} \leq H_k.$$

Let k be equal to 1. We obtain that

$$(H^0)^{(2)} = [[H^0, H^0], [H^0, H^0]] \leq [[H^0, H^0], H_0] \leq H_1.$$

Suppose the statement is true for k. Then we compute:

$$(H^0)^{(2k+2)} = [[(H^0)^{(2k)}, (H^0)^{(2k)}], [(H^0)^{(2k)}, (H^0)^{(2k)}]] \leq [[H_k, H_k], H^0] \leq H_{k+1}$$

This finishes the induction.

Hence $(H^0)^{(2n)}$ is a subgroup of the trivial group H_n, whence it is trivial as well and therefore H^0 is solvable of class at most $2n$. This can be expressed by a formula. So it is implied by finitely many of the formulas defining H^0. As H^0 is the intersection of a directed system definable subgroups, this also has to be true in one of those groups. Thus, one can find a definable solvable group of class at most $2n$ which has finite index in H and which is normal in G. □

Proposition 4.15. *Let G be a group definable in an NTP$_2$ theory, H be a normal solvable subgroup of G of class n and suppose that G is $|H|^+$-saturated. Then there exists a definable normal almost solvable subgroup S of G of class n containing H. Additionally, S admits a definable almost abelian series of length n such that all of its members are normal in G.*

Proof. We prove this by induction on the derived length n of H. If n is equal to 1 this is a consequence of the abelian case, Theorem 4.9(1). So let $n > 1$, and consider the abelian subgroup $H^{(n-1)}$ of H. It is a characteristic subgroup of H and hence, as H is normal in G, it is normal in G as well. So again by the abelian case, there exists a definable normal almost abelian subgroup A of G which contains $H^{(n-1)}$. Replacing G by G/A, we have that the derived length of HA/A is at most $n-1$ and we may apply the induction hypothesis which finishes the proof. □

Proof of Theorem 4.9(2). Applying Proposition 4.15 to the normal solvable subgroup H of G of class n gives us a definable almost solvable subgroup K of G of class n containing H which admits a definable almost abelian series of length n for which each member of the definable almost abelian series is normal in G. By Lemma 4.14, the group K has a definable subgroup S of finite index which is normal in G and solvable of class at most $2n$. □

Nilpotent subgroups

The following is an immediate consequence of Corollary 4.11

Corollary 4.16. *Let G be a group definable in an NTP$_2$ theory, let H be a subgroup of G and suppose that G is $|H|^+$-saturated. Then one can find definable subgroups A and K and a natural number m such that*

- *the cardinality of the conjugacy class k^A for all elements k in K is bounded by m;*
- *A is almost abelian and contains $Z(H)$;*
- *K contains H and A.*

If H is additionally normal in G, one can choose A and K to be normal in G as well.

Proof. Let $\bar{a}$, $\bar{b}$ and m be as in Corollary 4.11. Then $A = C_G(\bar{a}, \bar{b})$ and $K = C_G(\bar{b})$ are as required. Moreover, if H is additionally normal, letting $A = C_G(\bar{a}^G, \bar{b}^G)$ and $K = C_G(\bar{b}^G)$ gives the desired normal subgroups of G. □

Proof of Theorem 4.9(3). Note that if H is finite, the result holds trivially. So we may assume that H is infinite and so G is at least $\aleph_0$-saturated. For the second part of the theorem, we assume additionally that H is normal in G.

We prove by induction on the nilpotency class n of H that there exists a definable nilpotent subgroup N of G of class at most $2n$ and a sequence of subgroups:

$$\{1\} = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{2n} = N$$

such that $H \leq_v N$ and for all $0 \leq i < 2n$, we have that

- N_i is definable and normal in N;
- $[N_{i+1}, N] \leq N_i$.

If H is normal in G, we ask each N_i to be normal in G as well.

Let n be equal to 1. Then H is abelian, and by Theorem 4.9(1) there exists a definable almost abelian subgroup A of G which contains H. Note that the centralizer of any element of A has finite index in A. As the group $[A,A]$ is finite by Fact 3.23, we can put $N = N_2 = C_A([A,A])$ and $N_1 = Z([A,A])$. If H is normal in G, we may choose A to be normal in G as well. Since $C_A([A,A])$ and $Z([A,A])$ are characteristic subgroups of A, they are also normal in G which provides the second part of the theorem.

Now, let n (the nilpotency class of H) be strictly greater than 1 and assume that for any nilpotent subgroup of a definable group in an NTP$_2$ theory of class less than n, one can find a sequence as described above. The strategy is to find a definable subgroup N^* of G such that N^* virtually contains H and $Z_2(N^*)$ contains $N^* \cap Z(H)$. Then $(H \cap N^*)Z_2(N^*)/Z_2(N^*)$ has nilpotency class strictly smaller than n and we may apply the induction hypothesis. Thus, we would be able to find a definable nilpotent subgroup $N_{2n}/Z_2(N^*)$ of $G/Z_2(N^*)$ which virtually contains $N^*/Z_2(N^*)$ and therefore $H/Z_2(N^*)$. Taking the pullback to N^* together with its first and second center yields the desired properties.

We first show the following:

Claim. *There are definable subgroups A and K of G such that:*

- *A is a normal subgroup of K;*
- *$Z(H) \leq_v A$ and $H \leq K$;*
- *$K \leq \widetilde{C}_K(A)$*
- *$A \leq \widetilde{Z}(K)$;*
- *$[A,K]$ is finite and contained in $\widetilde{Z}(K)$.*

Proof. First, by Corollary 4.16 we can find definable subgroups A_0 and K of G and $m \in \omega$ such that

1. the cardinality of the conjugacy class k^{A_0} for all elements k in K is bounded by m;
2. A_0 is almost abelian and contains $Z(H)$;
3. K contains H and A_0.

The next step to prove the claim is to replace A_0 by a commensurable subgroup which is definable and additionally normal in K.

By (1) we deduce that for any element k in K, the index $[A_0 : C_{A_0}(k)]$ is bounded by m. So for k_0 and k_1 in K, we have that

$$[A_0^{k_0} : A_0^{k_1}] = [A_0 : A_0^{k_1 k_0^{-1}}] \leq [A_0 : C_{A_0}(k_1 k_0^{-1})] \leq m$$

Hence,

$$\mathcal{F} = \{{A_0}^k : k \in K\}$$

is a uniformly definable and uniformly commensurable family of subgroups of K. By Schlichting Theorem (Fact 4.8) one can find a definable subgroup A_1 of K which is commensurable with all groups in $\mathcal{F}$, in particular with A_0, and which is stabilized by all automorphisms which stabilize the family setwise. Thus A_1 is normal in K.

As A_1 is commensurable with A_0, we have that $K \leq \widetilde{C}_K(A_1)$. By symmetry of the almost centralizer (Corollary 3.14), we obtain that A_1 is virtually contained in $\widetilde{Z}(K)$, but A_1 need not be a subgroup of $\widetilde{Z}(K)$. Let $A = A_1 \cap \widetilde{Z}(K)$; this is still a normal subgroup of K and has finite index in A_1. Since the almost center of a definable group is not necessary definable in an NTP$_2$ theory, it is left to show that this intersection is indeed definable, as A and K satisfy all other properties of the claim (which will be explain in detail later).

Since A has finite index in A_1, the definable subgroup A_1 is a finite union of distinct cosets of A, say $A_1 = \bigcup_{i=1}^k a_i A$ for some $a_i \in A_1$. Furthermore, we have that A is the union of the definable sets

$$A_d := \phi_d(x) = \{x \in A_1 : [K : C_K(x)] < d\}.$$

But then we have that

$$A_1 = \bigcup_{i=1}^{k} \bigcup_{d \in \omega} a_i A_d$$

so by compactness and saturation of G this is equal to a finite subunion. Additionally, as $\{A_d\}_{d\in\omega}$ was a chain of subsets of A each contained in the next we have that

$$A_1 = \bigcup_{i=1}^{k} a_i A_d$$

for some fixed d. Hence A is equal to A_d and whence it is a definable normal subgroup of K. Moreover, the group A is commensurable with A_0, so it virtually contains $Z(H)$ and K is still contained in $\widetilde{C}_G(A)$. Additionally, A is contained in $\widetilde{C}_G(K)$ and normal in K. By Corollary 3.27, we have that the group $[A,K]$ is finite. As A is normal in K, we obtain that

$$[A,K] \leq A \leq \widetilde{Z}(K).$$

□

Let A and K be as in the claim. In particular, the index $[Z(H) : A \cap Z(H)]$ is finite. Take a set $H_0 := \{h_0, \ldots, h_n\}$ of representatives of each coset of $A \cap Z(H)$ in $Z(H)$, so that $Z(H) = h_0 (A \cap Z(H)) \cup h_1 (A \cap Z(H)) \cup \cdots \cup h_n (A \cap Z(H))$.

Let $K' := C_K(h_0, \ldots, h_n)$ and $A' := A \cap K'$.

Claim. *The following conditions hold:*

- $[A', K']$ *is finite and contained in* $\widetilde{Z}(K')$.
- $H \leq K'$.
- $Z(H) \cap A = Z(H) \cap A'$, *so that* $Z(H) \leq_v A'$.

Proof. We have that $Z(H) \cap A' \subseteq Z(H) \cap A$ and $[A', K'] \leq [A, K]$. Since $[A, K]$ is finite and contained in $\widetilde{C}_G(K)$, so is $[A', K']$. Furthermore, as K' is a subgroup of K, we have that $\widetilde{C}_G(K)$ is a subgroup of $\widetilde{C}_G(K')$. Moreover, since A' is a subgroup of K', the commutator group $[A', K']$ belongs to K', which yields the first item of the claim.

All of the h_i's in H_0 belong to $Z(H)$ and H is a subgroup of K, so $H \leq K' = C_K(h_0, \ldots, h_n)$.

Finally, let h be an element of $Z(H) \cap A$. We have that h belongs as well to K' and hence to A'. This completes the proof of the claim. □

Notice that in particular $Z(H) \cap A \leq A'$.

We can now define N^* as mentioned at the beginning of the proof. Let X be equal to $[A', K']$. Then we define:

$$N^* := C_{K'}(X).$$

Claim. *The following conditions hold:*

1. N^* *is a subgroup of* K' *of finite index, and thus* $H \cap N^*$ *has finite index in* H.
2. $Z(H) \cap N^* \leq Z_2(N^*)$.

Proof. Since X is contained in $\widetilde{Z}(K')$, the centralizer $C_{K'}(x)$ has finite index in K' for all x in X. As X is additionally finite, we obtain that N^* has finite index in K'. Since H is a subgroup of K', we have as well that $H \cap N^*$ has finite index in H, which proves (1).

To prove (2), observe first that since N^* is equal to $C_{K'}(X)$, we obtain immediately that $X \cap N^*$ is contained in $Z(N^*)$. Second, it is enough to show that

$$\{[z, n] : z \in Z(H) \cap N^*,\ n \in N^*\} \leq Z(N^*).$$

This will imply that $[Z(H) \cap N^*, N^*]$ is a subgroup of $Z(N^*)$ which yields that $Z(H) \cap N^*$ is contained in $Z_2(N^*)$.

As

$$Z(H) = \bigcup_{h_i \in H_0} h_i (A \cap Z(H)) = \bigcup_{h_i \in H_0} h_i (A' \cap Z(H)),$$

we can write z as a product of an element $h_i \in H_0$ and $a \in A'$. Thus

$$[z,n] = [h_i \cdot a, n] = [h_i, n]^a \cdot [a,n]$$

As n belongs to N^* which is a subgroup of $K' = C_K(H_0)$, the first factor is trivial and we obtain that:

$$[z,n] = [a,n] \in [A', K'] \leq X$$

Moreover, as z and n both belong to N^*, their commutator does as well. Thus we obtain finally that $[z,n]$ is an element of $X \cap N^*$ which is a subgroup of $Z(N^*)$ as shown above. So $Z(H) \cap N^*$ is contained in $Z_2(N^*)$ which finishes the claim. □

We are finally ready to prove the theorem, using the induction hypothesis. By the previous claim, we have that $Z(H) \cap N^* \leq Z_2(N^*)$. Hence

$$(H \cap N^*)/Z_2(N^*) \cap (H \cap N^*) \cong (H \cap N^*)\, Z_2(N^*)/Z_2(N^*)$$

is a quotient of $(H \cap N^*)/(Z\,(H) \cap N^*)$. We obtain that the nilpotency class of

$$(H \cap N^*)\, Z_2(N^*)/\, Z_2(N^*)$$

is at most the nilpotency class of $H/Z(H)$ which is strictly smaller than the one of H. Furthermore, it is contained in the group $N^*/Z_2(N^*)$ which is definable in an NTP_2 theory.

By induction hypothesis, we can find a sequence of subgroups of $N^*/Z_2(N^*)$

$$Z_2(N^*)/Z_2(N^*) \leq N_3/Z_2(N^*) \leq \cdots \leq N_{2n}/Z_2(N^*)$$

such that

$$(H \cap N^*)\, Z_2(N^*)/Z_2(N^*) \leq_v N_{2n}/Z_2(N^*)$$

and for all $2 \leq i \leq 2n$ we have that

- $N_i/Z_2(N^*)$ is definable and normal in $N_{2n}/Z_2(N^*)$;
- $[N_{i+1}, N_{2n}] \leq N_i$.

As N_{2n} is a subgroup of N^* we have that $Z(N^*) \cap N_{2n} \leq Z(N_{2n})$ and $[Z_2(N^*), N_{2n}] \leq Z(N^*)$. Note that the group $H \cap N^*$ is virtually contained in N_{2n} as well. As $H \cap N^*$ and H are commensurable, the same holds for H. So

$$\{1\} = N_0 \leq Z(N_{2n}) \leq Z_2(N_{2n}) \leq N_3 \cdots \leq N_{2n}$$

is an ascending central series of N_{2n} with the desired properties.

Now we treat the "moreover" part of Theorem 4.9(3). If H is normal in G, we may assume that K and A found in the first claim are normal in G. Thus, we can find definable normal subgroups A and K of G such that:

- $Z(H) \leq_v A$ and $H \leq K$;
- $K \leq \widetilde{C}_K(A)$
- $A \leq \widetilde{Z}(K)$;
- $[A, K]$ is finite and contained in $\widetilde{Z}(K)$.

Then we find as above a set of representatives $\{h_0, \ldots, h_n\}$ of the distinct cosets of $A \cap Z(H)$ in $Z(H)$ and we let K' be equal to $C_K(h_0^G, \ldots, h_n^G)$ and A' be equal to $K' \cap A$. These are both definable normal subgroups of G and we have as well that

- $[A', K']$ is finite and contained in $\widetilde{Z}(K')$.
- $H \leq K'$.
- $Z(H) \cap A = Z(H) \cap A'$, so that $Z(H) \leq_v A'$.

Now we define N^* to be the definable normal subgroup $C_{K'}([A', X'])$ which has by the above the following properties:

- N^* is a subgroup of K' of finite index, and thus $H \cap N^*$ has finite index in H.
- $Z(H) \cap N^* \leq Z_2(N^*)$.

The rest of the proof is exactly as the previous one, since our induction hypothesis now allows us to find all groups in the sequence to be normal in G. □

4.3 $\widetilde{\mathfrak{M}}_c$-groups

We want to prove the existence of definable envelopes for $\widetilde{\mathfrak{M}}_c$-group. For this, the crucial property of subgroups of $\widetilde{\mathfrak{M}}_c$-groups is that the iterated almost centralizers are definable.

Abelian groups

We first investigate the abelian case. It uses notions and results presented in Chapter 3. The proof is inspired by the one of the corresponding theorem for simple theories in [47].

Proposition 4.17. *Every almost abelian subgroup H of an $\widetilde{\mathfrak{M}}_c$-group is contained in a definable finite-by-abelian subgroup which is additionally normalized by $N_G(H)$.*

Proof. Let H be an almost abelian subgroup of the $\widetilde{\mathfrak{M}}_c$-group G and assume that G is $\aleph_0$-saturated. As G is an $\widetilde{\mathfrak{M}}_c$-group there are elements $h_0, \ldots, h_{n-1}$ in H and a natural number d such that for every element h in H, the index $[C : C \cap C_G(h)]$ is smaller than d for $C := \bigcap_{i=0}^{n-1} C_G(h_i)$. Observe additionally that H is virtually contained in C. Moreover, the following set

$$\mathcal{F} = \{C^h : h \in N_G(H)\}$$

is a family of uniformly commensurable definable subgroups of G. Thus applying Schlichting's theorem 4.8 to this family of subgroups, we obtain a definable subgroup D which is normalized by $N_G(H)$ and commensurable with C. So D virtually contains H and thus DH is a finite extension of D and thus definable. Note that:

- $\widetilde{Z}(DH)$ is a definable almost abelian group since DH is a definable subgroup of an $\widetilde{\mathfrak{M}}_c$-group and so $\widetilde{Z}(DH)$ coincides with the almost abelian group $\mathrm{FC}(DH)$.
- $H \leq \widetilde{Z}(DH)$ as DH is commensurable with C and thus the centralizer of any element of H has finite index in DH and as pointed out.
- $\widetilde{Z}(DH)$ is normalized by $N_G(H)$ as both D and H are.

So the definable almost abelian (thus finite-by-abelian) group $\widetilde{Z}(DH)$ contains H and is normalized by $N_G(H)$. □

Solvable groups

We prove first that any almost solvable subgroup of G is contained in a definable almost solvable subgroup which admits a definable almost abelian series. From this result, we deduce the existence of solvable envelopes up to finite index for almost solvable subgroups.

Proposition 4.18. *Let H be an almost solvable subgroup of class n of an $\widetilde{\mathfrak{M}}_c$-group G . Then there exists a definable almost solvable subgroup of class n which is normalized by $N_G(H)$ and admits a definable almost abelian series containing H.*

Proof. Let $\{1\} = H_0 \leq \cdots \leq H_n = H$ be an almost abelian series for H. We construct recursively a definable almost abelian series

$$\{1\} = S_0 \leq \cdots \leq S_n$$

such that for all $i \leq n$, we have that $H_i \leq S_i$ and S_i is normalized by $N_G(H)$.

As S_0 is the trivial group, we may suppose that S_{i-1} has been constructed for $0 < i < n$. Since S_{i-1} is definable and normalized by $N_G(H)$, we can replace G by the definable section $\mathbf{G}_i = N_G(S_{i-1})/S_{i-1}$. Note that this is an $\widetilde{\mathfrak{M}}_c$-group and that H_i/S_{i-1} is an almost abelian subgroup. Thus by the almost abelian case (Proposition 4.17), there exists a definable almost abelian subgroup $\mathbf{S}_i$ of $\mathbf{G}_i$ which is normalized by $N_{\mathbf{G}_i}(H_i/S_{i-1})$ containing H_i/S_{i-1}. As H_i is a characteristic subgroup of H and S_{i-1} is normalized by $N_G(H)$, the normalizer of H_i/S_{i-1} and thus of $\mathbf{S}_i$ contains $N_G(H)/S_{i-1}$. Now defining S_i to be the pullback of $\mathbf{S}_i$ in G, we conclude. □

Theorem 4.19. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be an almost solvable subgroup of class n. Then there exists a definable solvable group S of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H.*

Proof. We may assume that G is $\aleph_0$-saturated. Then, we can apply Proposition 4.18 to H which yields a definable almost solvable group K of class n containing H which admits a definable almost abelian series for which each member is normalized by $N_G(H)$. By Lemma 4.14, the group K has a definable subgroup S of finite index which is solvable of class at most $2n$ and which is normalized by $N_G(H)$. □

Nilpotent groups

Definition 4.20. A group H is *almost nilpotent* if there exists an *almost central series* of finite length, i. e. a sequence of normal subgroups of H

$$\{1\} \leq H_0 \leq H_1 \leq \cdots \leq H_n = H$$

such that H_{i+1}/H_i is a subgroup of $\mathrm{FC}(H/H_i)$ for every $i \in \{0, \ldots, n-1\}$. We call the least such $n \in \omega$, the *almost nilpotency class* of H.

Remark 4.21. The iterated FC-centers of any almost nilpotent group H of class n form an *almost central series* of length n.

In this section we prove that any almost nilpotent subgroup of class n is virtually contained in a definable nilpotent group of class at most $2n$. To do so, we need the following consequence of symmetry of the almost centralizer (Theorem 3.13) and Corollary 3.27.

Corollary 4.22. *Let G be a $\widetilde{\mathfrak{M}}_c$-group and H be an A-ind-definable subgroup of G. Then*

$$H \lesssim \widetilde{C}_G(\widetilde{C}_G(H))$$

Proof. Trivially, we have that $\widetilde{C}_G(H) \leq \widetilde{C}_G(H)$. As G is an $\widetilde{\mathfrak{M}}_c$-group and so $\widetilde{C}_G(H)$ is definable, we obtain the result using symmetry. □

Proposition 4.23. *Let G be an $\widetilde{\mathfrak{M}}_c$-group. Then the commutator $[\widetilde{Z}(G), \widetilde{C}_G(\widetilde{Z}(G))]$ is finite.*

Proof. We may assume that G is $\aleph_0$-saturated. As G is an $\widetilde{\mathfrak{M}}_c$-group, the normal subgroups $\widetilde{Z}(G)$ and $\widetilde{C}_G(\widetilde{Z}(G))$ are definable. As trivially $\widetilde{C}_G(\widetilde{Z}(G))$ is contained in itself and

$$\widetilde{Z}(G) = \widetilde{C}_G(G) \leq \widetilde{C}_G(\widetilde{C}_G(\widetilde{Z}(G))),$$

we may apply Corollary 3.27 to these two subgroups and obtain the result. □

Theorem 4.24. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and let H be an almost nilpotent subgroup of G of class n. Then there exists a definable nilpotent subgroup N of G of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H.*

Proof. We construct inductively on $i \leq n$ the following subgroups of G:
In the ith step we find a definable subgroup G_i of G and two definable normal subgroups N_{2i-1} and N_{2i} of G_i all normalized by $N_G(H)$ such that:

1. $H \leq_v G_i$;

2. $\mathrm{FC}_i(H) \cap G_i \leq N_{2i}$;

3. $[N_{2i-1}, G_i] \leq N_{2(i-1)}$;
4. $[N_{2i}, G_i] \leq N_{2i-1}$;
5. $G_i \leq G_{i-1}$.

Once the construction is done, we let N be equal to the definable group N_{2n} and consider the following sequence of definable subgroups:

$$\{1\} = N_0 \cap G_n \leq N_1 \cap G_n \leq \cdots \leq N_{2n} \cap G_n.$$

By the above, we have for all $j \leq 2n$ that

$$[N_j \cap G_n, N] \overset{(5)}{\leq} [N_j, G_{\lceil \frac{j}{2} \rceil}] \cap G_n \overset{(3)\text{ or }(4)}{\leq} N_{j-1} \cap G_n.$$

So N is a definable nilpotent subgroup of G of class at most $2n$ which is witnessed by the sequence above. Moreover N is normalized by $N_G(H)$ and

$$H = \mathrm{FC}_n(H) \overset{(1)}{\leq_v} \mathrm{FC}_n(H) \cap G_n \overset{(2)}{\leq} N.$$

Thus N virtually contains H. Hence, it remains to show the existence of such N_i's and G_i's.

Now, assume that $i > 0$ and that for $j < i$ and $k < 2i-1$ the groups N_k and G_j have been constructed. We work in the quotient $\mathbf{G} = G_{i-1}/N_{2(i-1)}$ which is an $\widetilde{\mathfrak{M}}_c$-group and we let $\mathbf{H} = (H \cap G_{i-1})/N_{2(i-1)}$ which is obviously normalized by $N_G(H)$. The first step is to replace $\mathbf{G}$ by a definable subgroup $\mathbf{C}$ which virtually contains $\mathbf{H}$ and such that $\mathrm{FC}_{\mathbf{G}}(\mathbf{H}) = \widetilde{Z}(\mathbf{C})$. Observe that the preimage of $\mathrm{FC}_{\mathbf{G}}(\mathbf{H})$ in G_{i-1} contains $\mathrm{FC}_i(H) \cap G_{i-1}$ as $\mathrm{FC}_{i-1}(H) \cap G_{i-1}$ is contained in $N_{2(i-1)}$.

If there is $g_0/N_{2(i-1)} \in \mathrm{FC}_{\mathbf{G}}(\mathbf{H}) \setminus \widetilde{Z}(\mathbf{G})$, we consider the family

$$\mathcal{H} = \{C_{\mathbf{G}}(g_0^h/N_{2(i-1)}) : h \in N_G(H)\}$$

Note that as $\mathbf{H}$ is normalized by $N_G(H)$, all members of $\mathcal{H}$ virtually contain $\mathbf{H}$. Moreover, as $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group there exists a finite intersection $\mathbf{F}$ of groups in $\mathcal{H}$ such that for any $\mathbf{K}$ in $\mathcal{H}$ we have that the index $[\mathbf{F} : \mathbf{F} \cap \mathbf{K}]$ is at most d. Thus the family

$$\{\mathbf{F}^h : h \in N_G(H)\}$$

is uniformly commensurable. So, by Schlichting's theorem (Fact 4.8) there is a definable subgroup $\mathbf{C}_0$ of $\mathbf{G}$ which is invariant under all automorphisms which stabilize the family setwise, thus normalized by $N_G(H)$, and commensurable with $\mathbf{F}$. Moreover $\mathbf{F} \cap \mathbf{H}$ is commensurable with $C_{\mathbf{H}}(g_0/N_{2(i-1)})$ as $g_0/N_{2(i-1)}$ belongs to $\mathrm{FC}_{\mathbf{G}}(\mathbf{H})$. Over all we obtain that

$$\mathbf{C}_0 \cap \mathbf{H} =_v \mathbf{H} \quad \text{and} \quad \mathbf{C}_0 \leq_v C_{\mathbf{G}}(g_0/N_{2(i-1)}). \quad (*)$$

If now, there is $g_1/N_{2(i-1)} \in \widetilde{C}_{\mathbf{C}_0}(\mathbf{H} \cap \mathbf{C}_0) \setminus \widetilde{Z}(\mathbf{C}_0)$, we can redo the same construction and obtain a $\mathbf{C}_1$. By $(*)$ and g_1 not belonging to $\widetilde{Z}(\mathbf{C}_0)$, we have that $C_{\mathbf{G}}(g_0/N_{2(i-1)}, g_1/N_{2(i-1)})$ has infinite index in $C_{\mathbf{G}}(g_0/N_{2(i-1)})$. Then we can iterated this process. It has to stop after finitely many steps, as for every j the index of $C_{\mathbf{G}}(g_0/N_{2(i-1)},\ldots,g_{j+1}/N_{2(i-1)})$ in $C_{\mathbf{G}}(g_0/N_{2(i-1)},\ldots,g_j/N_{2(i-1)})$ is infinite by construction, contradicting the fact that $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group. Letting $\mathbf{C}$ be equal to $\bigcap_i \mathbf{C}_i$, we found a definable subgroup of $\mathbf{G}$ (thus an $\widetilde{\mathfrak{M}}_c$-group), such that $\mathrm{FC}_{\mathbf{C}}(\mathbf{H}) = \widetilde{Z}(\mathbf{C})$. Additionally, the group $\mathbf{C}$ is normalized by $N_G(H)$ and its intersection with $\mathbf{H}$ has finite index in $\mathbf{H}$.

The next step is to define G_i, N_{2i-1} and N_{2i}. As $\mathbf{C}$ is an $\widetilde{\mathfrak{M}}_c$-group, Proposition 4.23 yields that the commutator $\mathbf{Z} = [\widetilde{Z}(\mathbf{C}), \widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))]$ is finite. Since $\widetilde{Z}(\mathbf{C})$ and $\widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))$ are characteristic subgroups of $\mathbf{C}$, we have that $\mathbf{Z}$ is normalized by $N_G(H)$ and contained in $\widetilde{Z}(\mathbf{C})$. Note additionally that the group $\widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))$ has finite index in $\mathbf{C}$ by Corollary 4.22. Thus $\mathbf{G}_i = \widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C})) \cap C_{\mathbf{C}}(\mathbf{Z})$ has finite index in $\mathbf{C}$. We let $\mathbf{N}_1 = \mathbf{Z} \cap \mathbf{G}_i$, a finite subgroup of the center of $\mathbf{G}_i$, and $\mathbf{N}_2 = \widetilde{Z}(\mathbf{C}) \cap \mathbf{G}_i = \widetilde{Z}(\mathbf{G}_i)$, which is contained in $Z(\mathbf{G}_i/\mathbf{N}_1)$. Note that all groups used to define $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are characteristic subgroups of $\mathbf{C}$ and thus $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are normalized by $N_G(H)$. Moreover, $\mathbf{N}_1$ and $\mathbf{N}_2$ are normal subgroups of $\mathbf{G}_i$. Let G_i, N_{2i-1} and N_{2i} be the preimages of $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ in G respectively. They satisfy all requirements, finishing the construction and therefore the proof. □

Corollary 4.25. *If H is a normal nilpotent subgroup of G of class n, there is a definable normal nilpotent subgroup of G that contains H of class at most $3n$.*

Proof. By the previous proposition, we can find a definable normal nilpotent subgroup N of G of class at most $2n$ that virtually contains H. Thus, the group HN is a finite union of cosets of the definable subgroup N in G. Therefore, we have that HN contains H and is a definable normal nilpotent subgroup which has nilpotency class at most $3n$ by Fitting's theorem (Fact 4.6). □

5 Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group

In this chapter we analyze the Fitting subgroup $F(G)$ (Definition 5.1) and the almost Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group. Note that $F(G)$ is always normal in G. Moreover, as the product of any two normal nilpotent subgroups is again nilpotent by Fitting's Theorem (Fact 4.6), we can conclude that $F(G)$ is locally nilpotent. It is even nilpotent if G is finite. On the other hand, if G is infinite its Fitting subgroup might not be nilpotent.

For $\mathfrak{M}_c$-groups, nilpotency of $F(G)$ was shown by Bryant [7] for G periodic, by Wagner [62] in the stable case and in general by Derakhshan and Wagner [15]. Furthermore, it has been recently generalized by Palacín and Wagner [50] to groups type-definable in simple theories. One of the main ingredients, other than the chain condition on centralizers, is that any nilpotent subgroup has a definable envelope up to finite index. As we establish this result for $\widetilde{\mathfrak{M}}_c$-groups in Section 4.3 we are able to prove nilpotency of the Fitting subgroup for $\widetilde{\mathfrak{M}}_c$-groups in this chapter. Afterwards, we analyze the approximate version of the Fitting group, which is the group generated by all normal almost nilpotent subgroups. We show that for $\widetilde{\mathfrak{M}}_c$-groups, this group is almost solvable. In the end, we analyze locally nilpotent $\widetilde{\mathfrak{M}}_c$-groups.

5.1 (Almost) Fitting subgroup

Let us first give the precise definition of the Fitting subgroup:

Definition 5.1. Let G be a group. The *Fitting subgroup* of G, denoted by $F(G)$, is the group generated by all normal nilpotent subgroups of G.

We make use of the following fact due to Ould Houcine:

Fact 5.2. *[28] For any $\aleph_0$-saturated group, nilpotency of the Fitting subgroup implies its definability.*

The first step is to show that any locally nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group, thus in particular the Fitting subgroup, is solvable.

Proposition 5.3. *Any locally nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is solvable.*

The proof is inspired by the corresponding result for type-definable groups in simple theories [50, Lemma 3.6]. For sake of completeness we give a detailed proof.

Proof. We may assume as usual that G is $\aleph_0$-saturated and thus any definable almost abelian group is a bounded almost abelian group.

Let K be a locally nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Let m be the minimal natural number such that each descending chain of intersection of centralizers in G with infinite indexes has length at most m. We consider all sequences of the form

$$G = C_G(g_1) > \cdots > C_G(g_1, \dots g_n)$$

such that each centralizer has infinite index in its predecessor and let $\mathcal{S}$ be the collection of such tuples $\bar{g} = (g_1, \dots, g_n)$. Note that n is at most m and that the first element of any tuple in $\mathcal{S}$ is an element of the center of G. We prove that $C_K(g_1, \dots g_{m-i})$ is solvable for any tuple $\bar{g} = (g_1, \dots g_{m-i})$ in $\mathcal{S}$ of length $m - i$ by induction on i.

For $i = 0$, the group $C_G(g_1, \dots g_m)$ is a definable almost abelian group. Using Fact 3.23 we obtain that its derived group is finite. As $C_K(g_1, \dots g_m)$ is a subgroup of $C_G(g_1, \dots g_m)$, its derived group is finite as well and additionally a subgroup of the locally nilpotent group K. Hence it is nilpotent and whence $C_K(g_1, \dots g_m)$ is solvable.

Now we assume that for any tuple in $\mathcal{S}$ of length at least $m-i$ the induction hypothesis holds. Let $\bar{g} = (g_1, \dots g_{m-i-1})$ be a tuple in $\mathcal{S}$ of length $m - (i + 1)$. We consider the group $C_K(g_1, \dots g_{m-i-1})$. By the induction hypothesis, we know that for any g in G for which $C_G(g)$ has infinite index in $C_G(g_1, \dots g_{m-i-1})$, the group $C_K(g_1, \dots g_{m-i-1}, g)$ is solvable. Therefore, letting H be equal to the locally nilpotent group $C_K(g_1, \dots g_{m-i-1})$ and replacing G by $C_G(g_1, \dots g_{m-i-1})$ (which is still an $\widetilde{\mathfrak{M}}_c$-group as it is a definable subgroup of an $\widetilde{\mathfrak{M}}_c$-group) yields that for any g such that $C_G(g)$ has infinite index in G, the centralizer $C_H(g)$ is solvable.

As $\widetilde{Z}(G)$ is a definable normal subgroup of the $\widetilde{\mathfrak{M}}_c$-group G, we can find some natural numbers n and d such that each descending chain of centralizers in G modulo $\widetilde{Z}(G)$ with index greater than d has length at most n.

If H is contained in the definable almost abelian group $\widetilde{Z}(G)$, the same argument as for i equal to 1 shows that H is solvable. Thus, we may suppose that H is not contained in the almost center of G. As H is locally nilpotent, we can find a nilpotent subgroup H_0 of H for which this holds, i. e. the group $H_0/\widetilde{Z}(G)$ is non-trivial. As H_0 is nilpotent, the group $Z(H_0/\widetilde{Z}(G))$ is non-trivial as well and hence $C_{H_0}(H_0/\widetilde{Z}(G))$ strictly contains $\widetilde{Z}(G) \cap H_0$. Take an element h_0 in their difference. If $C_H(h_0/\widetilde{Z}(G))$ has index greater than d in H, let $g_0, \dots, g_d$ in H be representative of distinct cosets and note that they are not contained in $\widetilde{Z}(G)$. As H is locally nilpotent, one can find a nilpotent subgroup H_1 of H containing $g_0, \dots, g_d$. Hence $C_{H_1}(h_0/\widetilde{Z}(G))$ has index greater than d in H_1 as well and $H_1/\widetilde{Z}(G)$ is non-trivial. Choose again an element h_1 in $C_{H_1}(H_1/\widetilde{Z}(G))\backslash\widetilde{Z}(G)$, so $C_H(h_1/\widetilde{Z}(G))$ contains H_1 and thus $C_H(h_0/\widetilde{Z}(G), h_1/\widetilde{Z}(G))$ has index greater than d in $C_H(h_1/\widetilde{Z}(G))$. If $C_H(h_1/\widetilde{Z}(G))$ has as well index greater than d in H we can iterate this process. By the choice of n and d this has to stop after at most n times and so we may find an element h in $H \setminus \widetilde{Z}(G)$ for which the group $C_H(h/\widetilde{Z}(G))$ has finite index in H. As h does not belong to the almost center of G, we have that $C_G(h)$ has infinite index in G and therefore $C_H(h)$ is solvable by assumption.

Let N be equal to the derived group of $\widetilde{C}_H(G) \leq \widetilde{Z}(G)$. Since it is finite and contained in H it is nilpotent. Consider the map from $C_H(h/N)$ to N sending x to $[h, x]$. This map has as kernel the solvable subgroup $C_H(h)$ and as image the nilpotent group N. So the subgroup $C_H(h/N)$ is solvable as well. The second step is to consider the map from $C_H(h/\widetilde{C}_H(G))$ to $\widetilde{C}_H(G)/N$ which maps x to $[h,x]/N$. Note that again the kernel $C_H(h/N)$ is solvable and the image $\widetilde{C}_H(G)/N$ is abelian. So $C_H(h/\widetilde{C}_H(G))$ is a solvable subgroup of finite index in H, say m. Using that G acts on G/H by left translation and that the kernel of the induced group morphism $\nu : G \to S_m$ is the intersection of all conjugates of H, we obtain that $\ker(\nu)$ is a normal subgroup of $C_H(h/\widetilde{C}_H(G))$ of finite index in H. As any finite quotient of a locally nilpotent group is nilpotent, the group H is solvable. This finishes the induction.

Taking a maximal tuple $(g_1, \ldots, g_m)$ in $\mathcal{S}$ and letting i be equal to $m-1$, we obtain that $C_K(g_1)$ is solvable. As K is equal to $C_K(g_1)$, this finishes the proof. □

Corollary 5.4. *The Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is solvable.*

In the next lemma we deal with a definable section of some $\widetilde{\mathfrak{M}}_c$-group acting via conjugation on another definable section. We recall and introduce some facts and notations:

Let G be a group that acts on an abelian group A by automorphisms. Then, one can naturally extend the action to the group ring $\mathbb{Z}[G]$, namely for an arbitrary element $\sum_{i<n} z_i g_i$ of $\mathbb{Z}[G]$ and a in A, we set

$$\left(\sum_{i<n} z_i g_i\right) \cdot a = \prod_{i<n} (g_i \cdot a)^{z_i}.$$

Moreover, we use the following **notation**:
If B is a subgroup of A and g an element of G we denote by $C_B(g)$ the group of elements b in B on which g acts trivially, i. e. $gb = b$. Furthermore, if H is a subgroup of G and a an element of A, we denote by $C_H(a)$ all elements h in H which act trivially on a. This yields the natural definition of an almost centralizer via this group action, namely for any subgroup B of A and H of G, we have that

$$\widetilde{C}_B(H) = \{b \in B : [H : C_H(b)] \text{ is finite}\}$$

$$\widetilde{C}_H(B) = \{h \in H : [B : C_B(h)] \text{ is finite}\}$$

Note that this group action defines a semidirect product $A \rtimes G$. Within this group, the above defined almost centralizer $\widetilde{C}_B(H)$ (respectively $\widetilde{C}_H(B)$) corresponds to the projection of $\widetilde{C}_{B\rtimes 1}(1 \rtimes H)$ to its first coordinate (respectively $\widetilde{C}_{1\rtimes H}(B \rtimes 1)$ to its second coordinate). So one obtains immediately the following symmetry for the above almost centralizer using Theorem 3.13 for $A \rtimes G$.

Lemma 5.5. *Let G be a group that acts on an abelian group A by automorphisms. Let H be a definable subgroup of G and B be a definable subgroup of A, then we have that*

$$H \lesssim \widetilde{C}_G(B) \text{ if and only if } B \lesssim \widetilde{C}_A(H).$$

Definition 5.6. Let G be a group and K, A, N and M be subgroups of G such that:

$$M \trianglelefteq K \quad \text{and} \quad N \trianglelefteq A.$$

We say that the quotient K/M *acts by conjugation on* A/N if the action by K/M on A/N via conjugation is well-defined, i. e.

- $K \leq N_G(A) \cap N_G(N)$;
- $M \leq C_G(A/N)$.

Lemma 5.7. *Let $\boldsymbol{K}$ and $\boldsymbol{A}$ be quotients of definable subgroups of an $\widetilde{\mathfrak{M}}_c$-group G such that $\boldsymbol{K}$ acts by conjugation on $\boldsymbol{A}$. Then the $\widetilde{C}_{\boldsymbol{K}}(\boldsymbol{A})$ and $\widetilde{C}_{\boldsymbol{A}}(\boldsymbol{K})$ are definable.*

Proof. The lemma is an immediate consequence of the following claim:

Claim. *There are natural numbers n and d (respectively n' and d') such that any descending chain of centralizers*

$$C_{\boldsymbol{A}}(\boldsymbol{k}_0) \geq C_{\boldsymbol{A}}(\boldsymbol{k}_0, \boldsymbol{k}_1) \geq \cdots \geq C_{\boldsymbol{A}}(\boldsymbol{k}_0, \dots, \boldsymbol{k}_m) \geq \dots \quad (\boldsymbol{k}_i \in \boldsymbol{K})$$

$$\Big(\textit{resp. } C_{\boldsymbol{K}}(\boldsymbol{a}_0) \geq C_{\boldsymbol{K}}(\boldsymbol{a}_0, \boldsymbol{a}_1) \geq \cdots \geq C_{\boldsymbol{K}}(\boldsymbol{a}_0, \dots, \boldsymbol{a}_m) \geq \dots \quad (\boldsymbol{a}_i \in \boldsymbol{A})\Big)$$

each of index greater than d (resp. d') in its predecessor is of length at most n (resp. n').

Proof of the claim. Suppose that the claim is false. Then, by compactness there exists an infinite descending chains of centralizer

$$C_{\mathbf{A}}(\mathbf{k}_0) \geq C_{\mathbf{A}}(\mathbf{k}_0, \mathbf{k}_1) \geq \cdots \geq C_{\mathbf{A}}(\mathbf{k}_0, \dots, \mathbf{k}_n) \geq \dots \quad (\mathbf{k}_i \in \mathbf{K})$$

$$\Big(\text{resp. } C_{\mathbf{K}}(\mathbf{a}_0) \geq C_{\mathbf{K}}(\mathbf{a}_0, \mathbf{a}_1) \geq \cdots \geq C_{\mathbf{K}}(\mathbf{a}_0, \dots, \mathbf{a}_m) \geq \dots \quad (\mathbf{a}_i \in \mathbf{A})\Big)$$

each of infinite index its predecessor. Let A, N, L and M be definable subgroups of G such that

$$\mathbf{A} = A/N \quad \text{and} \quad \mathbf{K} = K/M$$

and k_i in K such that $\mathbf{k}_i$ is equal to k_i/M as well as a_i in A such that $\mathbf{a}_i$ is equal to a_i/N. Then

$$\begin{aligned} C_{\mathbf{A}}(\mathbf{k}_i) &= \{a/N \in A/N : k_i/M \cdot a/N = a/N\} \\ &= \{a/N \in A/N : a^{k_i}/N = a/N\} \\ &= \{a \in A : ak_i/N = k_i a/N\}/N \\ &= C_A(k_i/N)/N \end{aligned}$$

$$\left(\begin{aligned} \text{resp. } C_{\mathbf{K}}(\mathbf{a}_i) &= \{k/M \in K/M : k/M \cdot a_i/N = a_i/N\} \\ &= \{k \in K : a_i^k/N = a_i/N\}/M \\ &= C_K(a_i/N)/M \end{aligned}\right)$$

Thus the above infinite descending chains of centralizer each of infinite index its predecessor translates to

$$C_A(k_0/N) \geq C_A(k_0/N, k_1/N) \geq \cdots \geq C_A(k_0/N, \dots, k_n/N) \geq \dots$$

$$\Big(\text{resp. } C_K(a_0/N) \geq C_K(a_0/N, a_1/N) \geq \cdots \geq C_A(a_0/N, \dots, a_n/N) \geq \dots \Big).$$

These are infinite descending chains of centralizer each of infinite index its predecessor in the definable section $N_G(N)/N$ of the $\widetilde{\mathfrak{M}}_c$-group G which is impossible. $\square_{\text{claim}}$

So, we can choose $\mathbf{k}_0,\dots,\mathbf{k}_n$ in $\widetilde{C}_{\mathbf{K}}(\mathbf{A})$ (resp. $\mathbf{a}_0,\dots,\mathbf{a}_{n'}$ in $\widetilde{C}_{\mathbf{A}}(\mathbf{K})$) such that for all $\mathbf{k}$ in $\widetilde{C}_{\mathbf{K}}(\mathbf{A})$ (resp. $\mathbf{a}$ in $\widetilde{C}_{\mathbf{A}}(\mathbf{K})$),

$$[C_{\mathbf{A}}(\mathbf{k}_0,\dots,\mathbf{k}_n) : C_{\mathbf{A}}(\mathbf{k}_0,\dots,\mathbf{k}_n,\mathbf{k}) < d]$$

$$\Big(\text{resp. } [C_{\mathbf{K}}(\mathbf{a}_0,\dots,\mathbf{a}_n) : C_{\mathbf{K}}(\mathbf{a}_0,\dots,\mathbf{a}_n,\mathbf{a}) < d']\Big).$$

Thus,

$$\widetilde{C}_{\mathbf{K}}(\mathbf{A}) = \Big\{\mathbf{k} \in \mathbf{K} : [C_{\mathbf{A}}(\mathbf{k}_0,\dots,\mathbf{k}_n) : C_{\mathbf{A}}(\mathbf{k}_0,\dots,\mathbf{k}_n,\mathbf{k}) < d]\Big\}$$

$$\Big(\text{resp. } \widetilde{C}_{\mathbf{A}}(\mathbf{K}) = \Big\{\mathbf{a} \in \mathbf{A} : [C_{\mathbf{K}}(\mathbf{a}_0,\dots,\mathbf{a}_n) : C_{\mathbf{K}}(\mathbf{a}_0,\dots,\mathbf{a}_n,\mathbf{a}) < d']\Big\}\Big)$$

□

The proof of [50, Lemma 3.8] which is stated for groups type-definable in a simple theory uses only symmetry of the almost centralizer and that they are definable. Hence it remains true for $\widetilde{\mathfrak{M}}_c$-groups.

Lemma 5.8. *Let $\boldsymbol{K}$ and $\boldsymbol{A}$ be definable sections of an $\widetilde{\mathfrak{M}}_c$-group G such that $\boldsymbol{A}$ is abelian and $\boldsymbol{K}$ acts by conjugation on $\boldsymbol{A}$. Suppose that $\boldsymbol{H}$ is an arbitrary abelian subgroup of $\boldsymbol{K}$ and that there are a tuple $\bar{\boldsymbol{h}} = (\boldsymbol{h}_i : i < \ell)$ in $\boldsymbol{H}$ and natural numbers $(m_i : i < \ell)$ such that*

- $(\boldsymbol{h}_i - 1)^{m_i}\boldsymbol{A}$ *is finite* $\forall i < \ell$;
- *for any $\boldsymbol{h}$ in $\boldsymbol{H}$ the index of $C_{\boldsymbol{A}}(\bar{\boldsymbol{h}},\boldsymbol{h})$ in $C_{\boldsymbol{A}}(\bar{\boldsymbol{h}})$ is finite.*

Then there is a definable subgroup $\boldsymbol{L}$ of $\boldsymbol{K}$ which contains $\boldsymbol{H}$ and a natural number m such that $\widetilde{C}^m_{\boldsymbol{A}}(\boldsymbol{L})$ has finite index in $\boldsymbol{A}$.

Proof. Let

$$\mathbf{L} = \widetilde{C}_{C_{\mathbf{K}}(\bar{\mathbf{h}})}(C_{\mathbf{A}}(\bar{\mathbf{h}})) = \{\mathbf{k} \in C_{\mathbf{K}}(\bar{\mathbf{h}}) : [C_{\mathbf{A}}(\bar{\mathbf{h}}) : C_{\mathbf{A}}(\bar{\mathbf{h}},\mathbf{k})] < \infty\}$$

with $\bar{\mathbf{h}}$ given by the statement (note that $C_{\mathbf{K}}(\bar{\mathbf{h}})$ denotes the centralizer within the group $\mathbf{K}$ and $C_{\mathbf{A}}(\bar{\mathbf{h}})$ denotes the centralizer given by the group action of $\mathbf{K}$ on $\mathbf{A}$). Observe that $\mathbf{L}$ contains $\mathbf{H}$ by assumption and that it is definable by Lemma 5.7.

Let m be equal to $1+\sum_{i=0}^{\ell-1}(m_i-1)$ and fix an arbitrary tuple $\bar{n} = (n_0,\dots,n_{m-1})$ in $\ell^{\times m}$. By the pigeonhole principle and the choice of m there is at least one i less than ℓ such that at least m_i many coordinates of $\bar{n}$ are equal to i. As the group ring $\mathbb{Z}(\mathbf{H})$ is commutative and $(\mathbf{h}_i - 1)^{m_i}\mathbf{A}$ is finite for all i less than ℓ by assumption, we have that

$$(\mathbf{h}_{n_0} - 1)(\mathbf{h}_{n_1} - 1)\dots(\mathbf{h}_{n_{m-1}} - 1)\mathbf{A}$$

is finite.

Claim. *Let $\boldsymbol{k}$ be an element of $\boldsymbol{K}$ and $\boldsymbol{B}$ be a subgroup of $\boldsymbol{A}$. Then we have that the set $(\boldsymbol{k}-1)\boldsymbol{B}$ is finite if and only if $\boldsymbol{B} \lesssim C_{\boldsymbol{A}}(\boldsymbol{k})$.*

Proof. Suppose that $\mathbf{B} \not\lesssim C_{\mathbf{A}}(\mathbf{k})$. Then there is a set of representatives $\{\mathbf{b}_i : i \in \omega\}$ of cosets of $\mathbf{B}$ modulo $C_{\mathbf{A}}(\mathbf{k})$, i. e. for i different than j we have that $\mathbf{b}_i - \mathbf{b}_j$ does not belong to $C_{\mathbf{A}}(\mathbf{k})$. Thus

$$(\mathbf{k} - 1)\mathbf{b}_i \neq (\mathbf{k} - 1)\mathbf{b}_j$$

which contradicts that $(\mathbf{k}-1)\mathbf{B}$ is finite.

On the other hand if $\mathbf{B} \lesssim C_{\mathbf{A}}(\mathbf{k})$ then there exists elements $\mathbf{b}_0,\ldots,\mathbf{b}_p$ in $\mathbf{B}$ such that for all $\mathbf{b}$ in $\mathbf{B}$ there exists i less or equal to p such that $\mathbf{b}-\mathbf{b}_i$ belongs to $C_{\mathbf{A}}(\mathbf{k})$, i. e. $(\mathbf{k}-1)\mathbf{b} = (\mathbf{k}-1)\mathbf{b}_i$. Hence the set $(\mathbf{k}-1)\mathbf{B}$ is equal to $(\mathbf{k}-1)\{\mathbf{b}_0,\ldots,\mathbf{b}_p\}$, whence finite. □

So, applying the claim to $\mathbf{B} = (\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)\mathbf{A}$, for all $i \le n$ we obtain that

$$(\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)\mathbf{A} \lesssim C_{\mathbf{A}}(\mathbf{h}_i).$$

Thus

$$(\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)\mathbf{A} \lesssim C_{\mathbf{A}}(\bar{\mathbf{h}}).$$

Since for all $\mathbf{k}_0$ in $\mathbf{L}$, we have that $C_{\mathbf{A}}(\bar{\mathbf{h}}) \lesssim C_{\mathbf{A}}(\mathbf{k}_0)$, we have as well that

$$(\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)\mathbf{A} \lesssim C_{\mathbf{A}}(\mathbf{k}_0)$$

and again by the claim we deduce that

$$(\mathbf{k}_0-1)(\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)\mathbf{A}$$

is finite. As $\mathbf{L}$ is contained in the centralizer of $\bar{\mathbf{h}}$, the previous line is equal to

$$(\mathbf{h}_{n_1}-1)\ldots(\mathbf{h}_{n_{m-1}}-1)(\mathbf{k}_0-1)\mathbf{A}.$$

We repeat the previous process m times and we obtain that for any m-tuple $(\mathbf{k}_0,\ldots\mathbf{k}_{m-1})$ in $\mathbf{L}$ we have that the set

$$(\mathbf{k}_{m-1}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A}$$

is finite. As the tuple is arbitrary, we have that for any $\mathbf{k}$ in $\mathbf{L}$ the group

$$(\mathbf{k}_{m-2}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A}$$

is almost contained in the centralizer $C_{\mathbf{A}}(\mathbf{k})$, i. e.

$$\mathbf{L} \le \widetilde{C}_{\mathbf{K}}((\mathbf{k}_{m-2}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A}).$$

By symmetry we have that

$$(\mathbf{k}_{m-2}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A} \lesssim \widetilde{C}_{\mathbf{A}}(\mathbf{L}).$$

By Lemma 5.7, we have that $\widetilde{C}_{\mathbf{A}}(\mathbf{L})$ is definable. Thus we may work modulo this group as $\mathbf{A}$ is abelian and obtain that

$$(\mathbf{k}_{m-2}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A}/\widetilde{C}_{\mathbf{A}}(\mathbf{L})$$

is finite for all choices of an $(m-1)$-tuple $(\mathbf{k}_0,\ldots,\mathbf{k}_{m-2})$ in $\mathbf{L}$. Thus as before we obtain by the claim and symmetry that

$$(\mathbf{k}_{m-3}-1)\ldots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A} \lesssim \widetilde{C}_{\mathbf{A}}(\mathbf{L}/\widetilde{C}_{\mathbf{A}}(\mathbf{L})) = \widetilde{C}^2_{\mathbf{A}}(\mathbf{L})$$

Repeating this process m times yields that $\mathbf{A} \lesssim \widetilde{C}^m_{\mathbf{A}}(\mathbf{L})$. □

Theorem 5.9. *The Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent and definable.*

Proof. Let G be an $\widetilde{\mathfrak{M}}_c$-group. Note first, that the Fitting subgroup $F(G)$ of G is solvable by Corollary 5.4. So there exists a natural number r such that the r^{th} derived subgroup $F(G)^{(r)}$ of $F(G)$ is trivial, hence nilpotent. Now we will show that if $F(G)^{(n+1)}$ is nilpotent, then so is $F(G)^{(n)}$. So, suppose that $F(G)^{(n+1)}$ is nilpotent. As it is additionally normal in G, using Corollary 4.25 we can find a definable normal nilpotent subgroup N of G containing $F(G)^{(n+1)}$. Moreover, note that the central series

$$\{1\} = N_0 < N_1 < \cdots < N_k = N$$

with $N_i = Z_i(N)$ consists of definable normal subgroups of G such that $[N, N_{i+1}] \leq N_i$.

Observe that it is enough to show that $F(G)^{(n)}$ is almost nilpotent: If $F(G)^{(n)}$ is almost nilpotent it has a normal nilpotent subgroup F of finite index by Theorem 4.24. As $F(G)^{(n)}$ is a subgroup of the Fitting subgroup, any finite subset is contained in a normal nilpotent subgroup of G. Thus, there is a normal nilpotent subgroup that contains a set of representatives of cosets of F in $F(G)^{(n)}$. Hence the group $F(G)^{(n)}$ is a product of two normal nilpotent subgroups, whence nilpotent by Fitting's Theorem (Fact 4.6).

As $F(G)^{(n)}/N$ is abelian and G/N is an $\widetilde{\mathfrak{M}}_c$-group, by Proposition 4.17 one can find a definable subgroup A' of G which contains $F(G)^{(n)}$ such that A'/N is an FC-group, i. e. $A' \leq \widetilde{C}_G(A'/N)$. Moreover, the group A'/N is normalized by the normalizer of $F(G)^{(n)}/N$ and thus A' is normal in G. The next step is to find a definable subgroup A of A' which still contains $F(G)^{(n)}$ and a natural number m for which $N \leq \widetilde{C}_G^m(A)$. This will imply that $A \leq \widetilde{C}_G(A/N) \leq \widetilde{C}_G(A/\widetilde{C}_G^m(A)) = \widetilde{C}_G^{m+1}(A)$. As A contains $F(G)^{(n)}$, the group $F(G)^{(n)}$ would be nilpotent by the above.

Fix now some $i > 0$. For any g in $F(G)^{(n)}$ there is some normal nilpotent subgroup H_g which contains g. So $N_i H_g$ is nilpotent by Fitting's theorem (Fact 4.6). Therefore, we can find a natural number m_g such that $[N_i \,,_{m_g} g] = \{1\}$ or seen with the group action as in Lemma 5.8

$$(g-1)^{m_g} N_i = \{1\}.$$

Additionally, as G is an $\widetilde{\mathfrak{M}}_c$-group, we can find a finite tuple $\bar{g}$ in $F(G)^{(n)}$ such that for any $g \in F(G)^{(n)}$ the index $[C_{N_i}(\bar{g}/N_{i-1}) : C_{N_i}(\bar{g}/N_{i-1}, g/N_{i-1})]$ is finite. So we may apply Lemma 5.8 to G/N acting on N_i/N_{i-1} and the abelian subgroup $F(G)^{(n)}/N$. Thus, there is a natural number m_i and a definable group K_i that contains $F(G)^{(n)}$ such that $N_i \lesssim \widetilde{C}_G^{m_i}(K_i/N_{i-1})$. Then the finite intersection $A = A' \cap \bigcap_i K_i$ is a definable subgroup of G which still contains $F(G)^{(n)}$. As for A', we have that $A \leq \widetilde{C}_G(A/N)$. Additionally:

$$N_i \lesssim \widetilde{C}_G^{m_i}(K_i/N_{i-1}) \leq \widetilde{C}_G^{m_i}(A/N_{i-1})$$

and inductively

$$\begin{aligned} N &\lesssim \widetilde{C}_G^{m_k}(A/N_{k-1}) \\ &\leq \widetilde{C}_G^{m_k}(A/(\widetilde{C}_G^{m_{k-1}}(A/N_{k-2}))) = C_G^{m_k+m_{k-1}}(A/N_{k-2}) \\ &\leq \ldots \leq C_G^{m_k+\cdots+m_1}(A) \end{aligned}$$

Using that $A \leq \widetilde{C}_G(A/N)$, we obtain that $A \leq \widetilde{C}_G^m(A)$ for $m = m_k + \cdots + m_1 + 1$.

Overall, we get that $F(G)^{(n)}$ is nilpotent for all n. In particular, the Fitting subgroup $F(G)$ of G is nilpotent. And finally by Fact 5.2 we deduce that it is definable as well. □

Now, we want to study the almost Fitting subgroup:

Definition 5.10. The *almost Fitting subgroup* of a group G is the group generated by all its normal almost nilpotent subgroups. We denote this subgroup by $\widetilde{F}(G)$.

Hickin and Wenzel show in [26] that the product of two normal almost nilpotent subgroups is again a normal almost nilpotent subgroup. Hence the almost Fitting subgroup of any group G is locally almost nilpotent but it might not be almost nilpotent. For $\widetilde{\mathfrak{M}}_c$-groups we show the following:

Proposition 5.11. *The almost Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is almost solvable.*

Proof. Let G be an $\widetilde{\mathfrak{M}}_c$-group and g be an element of its almost Fitting subgroup. Then there is a normal almost nilpotent subgroup H of G which contains g. By Theorem 4.24, we deduce that H has a nilpotent subgroup of finite index which is normal in G. Thus, the quotient $H/F(G)$ is finite. Since additionally H is a normal subgroup of G, we deduce that any element of H has finitely many conjugates modulo $F(G)$. Hence the group H and therefore $\widetilde{F}(G)$ are contained in $\widetilde{C}_G(G/F(G))$. As $F(G)$ is nilpotent by Theorem 5.9 and $\widetilde{C}_G(G/F(G))/F(G)$ is almost abelian, we deduce that $\widetilde{C}_G(G/F(G))$ is almost solvable. As any subgroup of an almost solvable group is almost solvable, we conclude that $\widetilde{F}(G)$ is almost solvable which finishes the proof. □

5.2 Locally nilpotent $\widetilde{\mathfrak{M}}_c$-groups

We finish this chapter with two proposition about locally nilpotent $\widetilde{\mathfrak{M}}_c$-group.

Proposition 5.12. *Let G be a locally nilpotent $\aleph_0$-saturated $\widetilde{\mathfrak{M}}_c$-group. Then G is nilpotent-by-finite.*

Proof. Note first of all, that it is enough to show that G is almost nilpotent as any almost nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent-by-finite by Theorem C(2).

As G is locally nilpotent, it is solvable by Proposition 5.3. So, we may inductively assume that G' is almost nilpotent. Thus G' is virtually contained in a definable normal nilpotent subgroup N of G by Theorem C(2). We claim that it is enough to show that for some natural number n, the group N is contained in $\widetilde{Z}_n(G)$: If so, we have that $G/\widetilde{Z}_n(G)$ is an almost abelian group as G/N is an almost abelian group and thus G is contained in $\widetilde{Z}_{n+1}(G)$.

Now, we prove inductively that for every natural number $i \leq m$, we can find a natural number j such that $Z_i(N)$ is contained in $\widetilde{Z}_j(G)$.

For i equals 0 this is trivially true. Thus, suppose that for $Z_i(N)$ we have found j such that $Z_i(N)$ is contained in $\widetilde{Z}_j(G)$. We work in $\mathbf{G} = G/\widetilde{Z}_j(G)$ which is again an $\widetilde{\mathfrak{M}}_c$-group. We set

$$\mathbf{N} := N\widetilde{Z}_j(G)/\widetilde{Z}_j(G) \quad \text{and} \quad \mathbf{N}_{i+1} := Z_{i+1}(N)\widetilde{Z}_j(G)/\widetilde{Z}_j(G).$$

As

$$[Z_{i+1}(N), N] \leq Z_i(N) \leq \widetilde{Z}_j(G),$$

we have that $[\mathbf{N}_{i+1}, \mathbf{N}] = 1$. Moreover, since G/N is an almost abelian group, so is $\mathbf{G}/\mathbf{N}$. We fix additionally the following notation:
For any subgroup $\mathbf{H}$ of $\mathbf{G}$, by $\mathbf{H}^*$ we denote $\mathbf{H}/\mathbf{N}$ and for any element $\mathbf{h}$ of $\mathbf{H}$ we write $\mathbf{h}^*$ for its class modulo $\mathbf{N}$. So, the group $\mathbf{G}^*$ acts on $\mathbf{N}_{i+1}$ by conjugation and we may regard $\mathbf{N}_{i+1}$ as an $\mathbf{G}^*$-module as $[N_{i+1}, N] = 1$.

Since $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group, we can find a finite tuple $\bar{\mathbf{g}} = (\mathbf{g}_0, \ldots, \mathbf{g}_m)$ of elements in $\mathbf{G}$ such that for any $\mathbf{g}$ in $\mathbf{G}$ the index $[C_{\mathbf{G}}(\bar{\mathbf{g}}) : C_{\mathbf{G}}(\bar{\mathbf{g}}, \mathbf{g})]$ is finite. Let $\mathbf{K}$ be equal to $C_{\mathbf{G}}(\bar{\mathbf{g}}/\mathbf{N})$ which has finite index in $\mathbf{G}$ as $\mathbf{G}/\mathbf{N}$ is almost abelian. For any $\mathbf{a} \in \mathbf{N}_{i+1}$, we have that the group generated by $\mathbf{a}$ and $\bar{\mathbf{g}}$ is a finitely generated subgroup of a locally nilpotent group and must be nilpotent. Thus for a given $\mathbf{a}$ in $\mathbf{N}_{i+1}$ there is a choice $\mathbf{h}_0, \ldots, \mathbf{h}_{d_a}$ of elements all belonging to the tuple $\bar{\mathbf{g}}$ such that

$$(\mathbf{h}_0^* - 1)(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_{d_a}^* - 1)\mathbf{a} = 0.$$

As $\mathbf{N}_{i+1}$ is definable and G is $\aleph_0$-saturated, there is an upper bound for the choice of d_a which we denote by d.

Thus, for any choice of $\mathbf{h}_0, \ldots, \mathbf{h}_d$ each being an element of the tuple $\bar{\mathbf{g}}$ and any element $\mathbf{a}$ of $\mathbf{N}_{i+1}$ we have in the right module notation

$$(\mathbf{h}_0^* - 1)(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)\mathbf{a} = 0.$$

As $\mathbf{a}$ was arbitrary in $\mathbf{N}_{i+1}$, we obtain that

$$(\mathbf{h}_0^* - 1)(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)\mathbf{N}_{i+1} = 0.$$

Moreover, since $\mathbf{h}_0$ is an arbitrary element of $\bar{\mathbf{g}}$, the previous equation yields that

$$(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)\mathbf{N}_{i+1} \leq C_{\mathbf{G}}(\bar{\mathbf{g}}).$$

Let $\mathbf{k}_0$ be any element of $\mathbf{K}$, by the choice of $\bar{\mathbf{g}}$, we obtain that

$$(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)\mathbf{N}_{i+1} \lesssim C_{\mathbf{G}}(\mathbf{k}_0)$$

or in other words

$$(\mathbf{k}_0^* - 1)(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)\mathbf{N}_{i+1} \quad \text{is finite.}$$

As $\mathbf{k}_0$ is an element of $C_{\mathbf{G}}(\bar{\mathbf{g}}/\mathbf{N})$ and $\mathbf{N}_{i+1}$ is commutative, this finite set equals

$$(\mathbf{h}_1^* - 1)\ldots(\mathbf{h}_d^* - 1)(\mathbf{k}_0^* - 1)\mathbf{N}_{i+1}$$

Iterating this process, we obtain that for any tuple of elements $(\mathbf{k}_0, \ldots, \mathbf{k}_d)$ in $\mathbf{K}$ we have that

$$(\mathbf{k}_d^* - 1)\ldots(\mathbf{k}_1^* - 1)(\mathbf{k}_0^* - 1)\mathbf{N}_{i+1} \quad \text{is finite.}$$

Since the tuple was taken arbitrary, we have that for any $\mathbf{k}$ in $\mathbf{K}$ the group

$$(\mathbf{k}_{d-1}^* - 1)\ldots(\mathbf{k}_1^* - 1)(\mathbf{k}_0^* - 1)\mathbf{N}_{i+1}$$

is almost contained in the centralizer $C_{\mathbf{N}_{i+1}}(\mathbf{k})$, i. e.

$$\mathbf{K} \leq \widetilde{C}_G((\mathbf{k}^*_{d-1}-1)\dots(\mathbf{k}^*_1-1)(\mathbf{k}^*_0-1)\mathbf{N}_{i+1})$$

By symmetry we have that

$$(\mathbf{k}^*_{d-1}-1)\dots(\mathbf{k}^*_1-1)(\mathbf{k}^*_0-1)\mathbf{N}_{i+1} \lesssim \widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$$

As $\mathbf{N}_{i+1}$ is an $\widetilde{\mathfrak{M}}_c$-group, the group $\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$ is definable, thus we may work modulo $\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$ and obtain that

$$(\mathbf{k}^*_{d-1}-1)\dots(\mathbf{k}^*_1-1)(\mathbf{k}^*_0-1)\mathbf{N}_{i+1}/\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$$

is finite for all choices of an $d-1$ tuple $(\mathbf{k}_0,\dots,\mathbf{k}_{m-2})$ in $\mathbf{K}$. Thus as before we obtain by symmetry that

$$(\mathbf{k}^*_{d-2}-1)\dots(\mathbf{k}^*_1-1)(\mathbf{k}^*_0-1)\mathbf{N}_{i+1} \lesssim \widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K}/\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})) = \widetilde{C}^2_{\mathbf{N}_{i+1}}(\mathbf{K}).$$

Repeating this process m many times yields that $\mathbf{N}_{i+1} \lesssim \widetilde{C}^d_{\mathbf{N}_{i+1}}(\mathbf{K}) = \widetilde{C}^d_{\mathbf{N}_{i+1}}(\mathbf{G}) \leq \widetilde{Z}_d(\mathbf{G})$. Thus $Z_{i+1}(N) \lesssim \widetilde{C}^d_G(G/\widetilde{Z}_j(G)) = \widetilde{Z}_{d+j}(G)$. As N and thus $Z_{i+1}(N)$ are normal in G, this yields immediately that $Z_{i+1}(N) \leq \widetilde{Z}_{d+j+1}(G)$ which finishes the proof. □

Proposition 5.13. *Let G be a locally nilpotent $\widetilde{\mathfrak{M}}_c$-group such that $G/\widetilde{Z}_k(G)$ has finite exponent for some natural number k. Then G is nilpotent-by-finite.*

Proof. First of all note, that it is enough to show that $G/\widetilde{Z}_k(G)$ is almost nilpotent, as this implies that G is almost nilpotent and any almost nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent-by-finite by Theorem 4.24. So let us replace G by $G/\widetilde{Z}_k(G)$ which is as well an $\widetilde{\mathfrak{M}}_c$-group by definition, locally nilpotent and of finite exponent.

The rest of the proof is analogous to the previous one. Using the same notation as before, the only difference is the way to find the bound d such that for any choice of $h_0,\dots,h_d$ each being an element of the tuple $\bar{g}$ and any element of $\mathbf{N}_{i+1}$ we have in the right module notation that

$$(h^*_0-1)(h^*_1-1)\dots(h^*_d-1)a = 0.$$

In this context, we know that $\mathbf{G}$ has finite exponent, say e. Thus, the group generated by $\bar{g}$ has finite order, say f. So for any $a \in \mathbf{N}_{i+1}$, the group generated by a and $\bar{g}$ has order at most $d = e^f \cdot f$ and as it is a finitely generated subgroup of a locally nilpotent group, it is nilpotent. Thus it is nilpotent of class at most d which gives the bound. □

6 Almost commutator and almost nilpotent subgroups

In Chapter 3 we introduced the almost centralizer which is a centralizer "up to finite index". Thus one might ask, if there exists a corresponding notion of an "almost commutator". The main goal is to introduce such a notion and to establish its basic properties. Even though this notion might not have the desired properties in the general context, it has once we work in $\widetilde{\mathfrak{M}}_c$-groups. This allows us to generalize a result on nilpotent subgroups to almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups.

For the rest of the chapter we fix a parameter set A and let G be an $|A|^+$-saturated and $|A|^+$-homogeneous group.

6.1 Almost commutator

To simplify the notation in the next definition, we let $\mathcal{G}$ be the family of all A-definable subgroups of G. Note that this family is stable under finite intersections.

Definition 6.1. For two A-ind-definable subgroups H and K of G, we define:

$$\tilde{[}H,K\tilde{]}_A := \bigcap\{L \in \mathcal{G} : L = L^{N_G(H)} = L^{N_G(K)}, H \lesssim \widetilde{C}_G(K/L)\}$$

and call it the *almost A-commutator* of H and K. If A is the empty set we omit the index and just say the almost commutator.

By Theorem 3.13 the almost commutator is symmetric, i. e. for two A-ind-definable subgroups H and K, we have $\tilde{[}H,K\tilde{]}_A = \tilde{[}K,H\tilde{]}_A$. Moreover, it is the intersection of definable subgroups of G. Note that the ordinary commutator of two A-ind-definable groups is not necessary definable nor the intersection of definable subgroups, and hence one cannot compare it with its approximate version, contrary to the almost centralizer.

Observe that the final results we obtain in this section only deal with normal subgroups of $\widetilde{\mathfrak{M}}_c$-groups. Thus, we **restrict our framework** from now on **to normal subgroups**. In this case, namely given two normal A-ind-definable subgroups H and K of G, the subgroup $\tilde{[}H,K\tilde{]}$ is the intersection of normal subgroups of G which simplifies not only the definition but also many arguments and ambiguities in numerous proofs. Note anyhow that all results in this section could be generalized to arbitrary subgroups.

So let from now on let $\mathcal{F}$ be the family of all A-definable normal subgroups of G. Note that this family is still stable under finite intersections and additionally under finite products.

Then the definition of the almost commutator of two ind-definable normal subgroups H and K of G simplifies to:

$$\widetilde{[H,K]}_A := \bigcap \{L \in \mathcal{F} : H \lesssim \widetilde{C}_G(K/L)\}.$$

As $H \lesssim \widetilde{C}_G(K/L)$ does not depend on the model we choose, the almost commutator does not depend on G. In other words, in any elementary extension of G, it will correspond to the intersection of the same A-definable groups.

In the rest of this section, we establish basic properties of the almost commutator of ind-definable normal subgroups in arbitrary groups. To simplify notation, **we add A as constants to the language** and thus for any two A-ind-definable subgroups H and K of G, **the almost commutator $\widetilde{[H,K]}$ and the A-almost commutator $\widetilde{[H,K]}_A$ in the new language coincide**. Therefore, we may omit A in the index in the rest of the section.

For two A-ind-definable normal subgroups H and K of G and L the intersection of A-definable normal subgroups of G, we obtain immediately that

$$H \lesssim \widetilde{C}_G(K/L) \quad \text{implies} \quad \widetilde{[H,K]} \leq L.$$

The other implication is a consequence of the following result:

Lemma 6.2. *For any A-ind-definable normal subgroups H and K of G, we have that*

$$H \lesssim \widetilde{C}_G\Big(K \Big/ \widetilde{[H,K]}\Big).$$

Moreover, $\widetilde{[H,K]}$ is the smallest intersection of A-definable normal subgroups for which this holds.

Proof. We let $\mathcal{L}$ be the family of all A-definable normal subgroups L of G such that $H \lesssim \widetilde{C}_G(K/L)$. Suppose that $H \not\lesssim \widetilde{C}_G(K/\widetilde{[H,K]})$. As $\widetilde{[H,K]}$ is the intersection of the normal subgroups L in $\mathcal{L}$, Properties 3.8 (10) yields that there is an L in $\mathcal{L}$ such that $H \not\lesssim \widetilde{C}_G(K/L)$. This contradicts the choice of $\mathcal{L}$ and the first part of the lemma is established.

Now, let L be an intersection of A-definable normal subgroups such that $H \lesssim \widetilde{C}_G(K/L)$. Then, this holds for any of the definable subgroups in the intersection. Thus, those subgroups contain $\widetilde{[H,K]}$ and therefore L contains $\widetilde{[H,K]}$. □

Using the previous lemma we obtain immediately the following corollaries.

Corollary 6.3. *Let H and K be two A-ind-definable normal subgroups of G and L be an intersection of A-definable normal subgroups of G. Then, we have that $H \lesssim \widetilde{C}_G(K/L)$ if and only if $\widetilde{[H,K]} \leq L$.*

Corollary 6.4. *For any almost commutator of two A-ind-definable normal subgroups H and K and any intersection L of A-definable normal subgroups, we have that $\widetilde{[H,K]} \lesssim L$ if and only if $\widetilde{[H,K]} \leq L$.*

Proof. The implication from right to left is trivial. So suppose that $\widetilde{[H,K]} \lesssim L$. Lemma 6.2 yields that $H \lesssim \widetilde{C}_G(K/\widetilde{[H,K]})$. Furthermore, by assumption we have that the intersection of A-definable subgroups $\widetilde{[H,K]} \cap L$ has bounded index in $\widetilde{[H,K]}$, i. e. we have that $\widetilde{[H,K]} \cap L \sim \widetilde{[H,K]}$. So Properties 3.8 (7) yields that $H \lesssim \widetilde{C}_G(K/(\widetilde{[H,K]} \cap L))$. As $\widetilde{[H,K]}$ is the smallest subgroup for which this holds, we obtain the result. $\square$

The next lemma seems rather trivial but it is essential for almost any proof concerning computations with almost commutators.

Lemma 6.5. *Let H, K, N and M be A-ind-definable normal subgroups of G.*

1. *If $N \lesssim H$ and $M \lesssim K$ then $\widetilde{[N,M]} \leq \widetilde{[H,K]}$.*
2. *If H (resp. K) is an intersection of definable groups $\widetilde{[H,K]}$ is contained in H (resp. K).*

Proof. 1. Let L be an arbitrary A-definable normal subgroup of G such that H is almost contained in $\widetilde{C}_G(K/L)$. Since $K \cap M$ is a subgroup of K, we have that H is almost contained in $\widetilde{C}_G(K \cap M/L)$ as well. As N is almost contained in H, we may replace H by N and obtain that N is almost contained in $\widetilde{C}_G(K \cap M/L)$. Additionally, the almost centralizer of two commensurate A-ind-definable subgroups such as M and $K \cap M$ coincides. Thus we conclude that N is almost contained in $\widetilde{C}_G(M/L)$ or in orther words $\widetilde{[N,M]}$ is a subgroup of L. As L was arbitrary, the almost commutator $\widetilde{[N,M]}$ is contained in $\widetilde{[H,K]}$.

2. We have trivially that $H \leq \widetilde{C}_G(K/H)$. So if H is the intersection of definable groups, we conclude that the almost commutator of H and K is contained in H.

$\square$

Lemma 6.6. *Let H and K be two A-type-definable normal subgroups of an $|A|^+$-saturated group G. Fix $\{H_i : i \in I\}$ and $\{K_s : s \in S\}$ two projective systems of A-definable sets such that $H = \bigcap_{i \in I} H_i$ and $K = \bigcap_{s \in S} K_s$ (i. e. for any i, j in I and s, t in S there exists n in I and m in S such that $H_i \cap H_j \supseteq H_n$ and $K_s \cap K_t \supseteq K_m$). Then, we have that*

$$HK = \bigcap_{(i,s) \in I \times S} H_i K_s.$$

Proof. Inclusion from left to right is obvious. So take c in $\bigcap_{(i,s) \in I \times S} H_i K_s$. Thus for all distinct i and I and s in S there exists elements h_i of H_i and k_s of K_s such that c is equal to $h_i k_s$. So the following type over A is consistent.

$$\pi(x,y) = \{x \in H_i : i \in I\} \cup \{y \in K_s : s \in S\} \cup \{c = xy\}$$

By compactness and saturation of G, one can find $h \in \bigcap_{i \in I} H_i = H$ and $k \in \bigcap_{s \in S} K_s = K$ such that $c = hk$. $\square$

Lemma 6.7. *Let H, K, and L be A-ind-definable normal subgroups of G. Then we have*

$$\widetilde{[HK,L]} \leq \widetilde{[H,L]} \cdot \widetilde{[K,L]}.$$

Proof.

$$\begin{aligned} \widetilde{[H,L]} \cdot \widetilde{[K,L]} &= \bigcap \{M \in \mathcal{F} : H \lesssim \widetilde{C}_G(L/M)\} \cdot \bigcap \{N \in \mathcal{F} : K \lesssim \widetilde{C}_G(L/N)\} \\ &\overset{6.6}{=} \bigcap \{M \cdot N : M, N \in \mathcal{F}, H \lesssim \widetilde{C}_G(L/M), K \lesssim \widetilde{C}_G(L/N)\} \end{aligned}$$

As the product of two groups in $\mathcal{F}$ is again a subgroup which belongs to $\mathcal{F}$, for M and N in $\mathcal{F}$ such that $H \lesssim \widetilde{C}_G(L/M)$ and $K \lesssim \widetilde{C}_G(L/N)$, by Properties 3.8 we have that $H \lesssim \widetilde{C}_G(L/MN)$ and $K \lesssim \widetilde{C}_G(L/MN)$. So by Lemma 3.5 we obtain $HK \lesssim \widetilde{C}_G(L/MN)$. Thus, the previous set contains the following one:

$$\begin{aligned} &\supseteq \bigcap \{P \in \mathcal{F} : HK \lesssim \widetilde{C}_G(L/P)\} \\ &= [HK, L]^{\sim} \end{aligned}$$

This finishes the proof. $\square$

Another useful interaction between the almost centralizer and the almost commutator is the following:

Lemma 6.8. *Let H and K be two A-ind-definable normal subgroups of G and L be an intersection of A-definable normal subgroups of G. If $[H,K]^{\sim} \leq L$ then $H \leq \widetilde{C}^2_G(K/L)$.*

Proof. Let $[H,K]^{\sim}$ be contained in L. By Corollary 6.3, we have that $H \lesssim \widetilde{C}_G(K/L)$. So $H/\widetilde{C}_G(K/L)$ is a bounded group and as H is normal in G, it contains $h^k \cdot \widetilde{C}_G(K/L)$ for all h in H and k in K. Hence the set $\{h^k : k \in K\}/\widetilde{C}_G(K/L)$ of conjugates of any element h in H by K modulo $\widetilde{C}_G(K/L)$ is bounded. As the size of this set corresponds to the index of $C_K(h/\widetilde{C}_K(K/L))$ in K, the group H is contained in the almost centralizer $\widetilde{C}_G(K/\widetilde{C}_K(K/L))$, i. e. the group H is contained in $\widetilde{C}^2_G(K/L)$. $\square$

We would like to translate the approximate version of the three subgroups lemma into the notation of almost commutators. The problem we are facing is that the almost centralizer of a subgroup is not necessarily definable. This leads to our next section, where we investigate normal nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups making use of the almost commutator.

6.2 Almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups

A consequence of the definability of the almost centralizer in $\widetilde{\mathfrak{M}}_c$-groups (Proposition 3.28) is that the almost commutator is "well behaved". For example, we obtain the lemma below:

Lemma 6.9. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be an A-ind-definable normal subgroup of G. For any natural number n, we have that*

$$[H, \widetilde{C}^n_G(H)]^{\sim} \leq \widetilde{C}^{n-1}_G(H)$$

Proof. We have that

$$[H, \widetilde{C}^n_G(H)]^{\sim} = \left[H, \widetilde{C}_G\left(H \middle/ \widetilde{C}^{n-1}_G(H)\right)\right]^{\sim}$$

by definition of the almost centralizer. Moreover, the almost centralizer $\widetilde{C}^{n-1}_G(H)$ is an A-definable subgroup of G since G is an $\widetilde{\mathfrak{M}}_c$-group. Thus

$$\left[H, \widetilde{C}_G\left(H \middle/ \widetilde{C}^{n-1}_G(H)\right)\right]^{\sim} \leq \widetilde{C}^{n-1}_G(H)$$

as $\widetilde{C}_G(H/\widetilde{C}^{n-1}_G(H))$ is trivially contained in itself and we obtain the result. $\square$

The main goal is to show a version of Hall nilpotency criteria for almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups. The ordinary version is the following:

Fact 6.10. *[21, Theorem 7] Let N be normal subgroup of G. If N is nilpotent of class m and $G/[N,N]$ is nilpotent of class n then G is nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2}$.*

We first have to state the approximate three subgroups lemma in terms of the almost commutator.

Notation. Let H, K and L be A-ind-definable normal subgroups of a given group G. Recall that for the ordinary commutator, we write $[H,K,L]$ for $[[H,K],L]$. Similarly, for the almost commutator, we write $\tilde{[}H,K,L\tilde{]}$ for $\tilde{[}\tilde{[}H,K\tilde{]},L\tilde{]}$. Note that the group $\tilde{[}H,K\tilde{]}$ is an A-ind-definable normal subgroup of G and thus $\tilde{[}\tilde{[}H,K\tilde{]},L\tilde{]}$ is well defined.

Now, given an $\widetilde{\mathfrak{M}}_c$-group G, we have that the almost centralizer of any A-ind-definable subgroup in G is definable. Thus for H, K and L such that H and K normalize L, we have that $H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$ if and only if $\tilde{[}H,K\tilde{]} \leq \widetilde{C}_G(L)$ by Corollary 6.3. This again is equivalent to $\tilde{[}H,K,L\tilde{]}$ being trivial. With this equivalence, we may phrase Theorem 3.19 for $\widetilde{\mathfrak{M}}_c$-groups as below.

Corollary 6.11. *Let H, K and L be three A-ind-definable strongly normal subgroups of an $\widetilde{\mathfrak{M}}_c$-group G. Then for any M which is an intersection of A-definable normal subgroups of G, we have that*

$$\tilde{[}H,K,L\tilde{]} \leq M \text{ and } \tilde{[}K,L,H\tilde{]} \leq M \text{ imply } \tilde{[}L,H,K\tilde{]} \leq M.$$

Proof. Let M be equal to the intersection of definable normal subgroups M_i with $i < \kappa$. For any i less than κ, we may work in the group G modulo M_i which is a quotient of an $\widetilde{\mathfrak{M}}_c$-group by a definable normal subgroup and so an $\widetilde{\mathfrak{M}}_c$-group as well. Hence, Theorem 3.19 (working modulo the definable group M_i) yields that

$$H \lesssim \widetilde{C}_G\left(K\middle/\widetilde{C}_G(L/M_i)\right) \text{ and } K \lesssim \widetilde{C}_G\left(L\middle/\widetilde{C}_G(H/M_i)\right)$$

imply

$$L \lesssim \widetilde{C}_G\left(H\middle/\widetilde{C}_G(K/M_i)\right).$$

We can translate this to

$$\tilde{[}H,K,L\tilde{]} \leq M_i \text{ and } \tilde{[}K,L,H\tilde{]} \leq M_i \text{ imply } \tilde{[}L,H,K\tilde{]} \leq M_i$$

So the statement is true for any M_i and hence for the intersection. □

Now, we want to define the notion of an *almost lower central series* and find a characterization of being almost nilpotent via this series.

In literature the ordinary lower central series of a subgroup H of G is defined as follows:

$$\gamma_1 H = H \quad \text{and} \quad \gamma_{i+1} H = [\gamma_i H, H].$$

Analogously, we introduce the following:

Definition 6.12. We define the *almost lower A-central series* of an A-ind-definable subgroup H of G as follows:

$$(\widetilde{\gamma}_1 H)_A = H \quad \text{and} \quad (\widetilde{\gamma}_{i+1} H)_A = [\widetilde{\gamma}_i H, H]^{\sim}_A.$$

We also refer to $(\widetilde{\gamma}_n H)_A$ as the *iterated n^{th} almost commutator* of H. Again, if A is the empty set we omit the index.

As we have added A as constants to the language, we may omit it again in the subscript of the iterated n^{th} almost commutator for the rest of the chapter.

Remark 6.13. The almost lower center series is well-defined as $[H,H]^{\sim}$ is the intersection of A-definable groups and hence A-type-definable. Thus, by induction we see that $\widetilde{\gamma}_{i+1} H = [\widetilde{\gamma}_i H, H]^{\sim}$ is again an A-type-definable subgroup.

To make the proofs more readable, we fix the following notation:

Notation. If $K_1, \ldots, K_n$ are A-ind-definable subgroups of G, let

$$\widetilde{\gamma}_n(K_1, \ldots, K_n) := [\ldots[[K_0, K_1]^{\sim}, K_2]^{\sim}, \ldots, K_n]^{\sim}.$$

If $K_i, \ldots, K_{i+j-1}$ are all equal to K we can replace the sequence by K^j, namely write $\widetilde{\gamma}_n(K_1, \ldots, K_n)$ as $\widetilde{\gamma}_n(K_1, \ldots K_{i-1}, K^j, K_{i+j}, \ldots, K_n)$. Also,

$$\widetilde{\gamma}_{i+0+j}(K_1, \ldots K_i, K^0, K_{i+1}, \ldots, K_{i+j}) = \widetilde{\gamma}_{i+j}(K_1, \ldots K_i, K_{i+1}, \ldots, K_{i+j}).$$

Observe that $\widetilde{\gamma}_n(H^n)$ is another way of writing $\widetilde{\gamma}_n H$.

We want to establish a connection between the triviality of the nth iterated almost commutator of a normal subgroup H of G and the almost nilpotency class of H.

Lemma 6.14. *If H is an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G and almost nilpotent of class n, then $\widetilde{\gamma}_{n+1} H$ is trivial. Conversely, if $\widetilde{\gamma}_{n+1} H$ is trivial, then H is almost nilpotent of class at most $n+1$.*

Proof. To prove the first result, we show by induction on $i \leq n$ that the almost commutator $\widetilde{\gamma}_{i+1} H$ is contained in $\widetilde{C}_G^{n-i}(H)$. As H is almost nilpotent of class n, i. e. $H \leq \widetilde{C}_G^n(H)$, the inclusion is satisfied for i equals to zero. Now suppose it holds for all natural numbers smaller or equal to i. The induction hypothesis together with Lemma 6.5(1) implies that $\widetilde{\gamma}_{i+2} H = [\widetilde{\gamma}_{i+1} H, H]^{\sim}$ is contained in $[\widetilde{C}_G^{n-i}(H), H]^{\sim}$. Moreover, by Lemma 6.9 we have that $[\widetilde{C}_G^{n-i}(H), H]^{\sim}$ is contained in $\widetilde{C}_G^{n-i-1}(H)$. Hence $\widetilde{\gamma}_{i+2} H$ is also contained in $\widetilde{C}_G^{n-i-1}(H)$ which finishes the induction. Letting i be equal to n, we obtain that $\widetilde{\gamma}_{n+1} H$ is contained in $\widetilde{C}_G^0(H)$ which is the trivial group by definition.

For the second result, we first show the following inclusion by induction that for i less or equal to $n-1$:

$$\widetilde{\gamma}_{(n+1)-i} H \leq \widetilde{C}_G^i(H).$$

For $i = 0$, the inequality holds by hypothesis. Now we assume, the inequality holds for $i < n-1$. Thus $\widetilde{\gamma}_{(n+1)-i}H \leq \widetilde{C}^i_G(H)$ or in other words $[\widetilde{\gamma}_{(n+1)-(i+1)}H, H]^{\sim} \leq \widetilde{C}^i_G(H)$. By Corollary 6.3, we have that

$$\widetilde{\gamma}_{(n+1)-(i+1)}H \lesssim \widetilde{C}_G\left(H \big/ \widetilde{C}^i_G(H)\right) = \widetilde{C}^{i+1}_G(H).$$

By Corollary 6.4, as $(n+1)-(i+1)$ is at least 2, finally we obtain $\widetilde{\gamma}_{(n+1)-(i+1)}H \leq \widetilde{C}^{i+1}_G(H)$ which finishes the induction.

Now, we let i be equal to $n-1$ and we obtain: $[H,H]^{\sim} \leq \widetilde{C}^{n-1}_G(H)$. Then by Lemma 6.8 we have that $H \leq \widetilde{C}^{n+1}_G(H)$ and hence H is almost nilpotent of class $n+1$. □

The next three lemmas are the preparation to finally show the approximate version of Hall's nilpotency criteria.

Lemma 6.15. *Let N be a normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then for all positive natural numbers n and m, we have that*

$$[\widetilde{\gamma}_n N, \widetilde{\gamma}_m N]^{\sim} \leq \widetilde{\gamma}_{n+m} N.$$

Proof. We proof this by induction on $m > 0$.

If m is equal to 1, we have immediately that for all $n > 0$,

$$[\widetilde{\gamma}_n N, \widetilde{\gamma}_1 N]^{\sim} \leq [\widetilde{\gamma}_n N, N]^{\sim} \leq \widetilde{\gamma}_{n+1} N.$$

To continue the induction, suppose that for a given $m > 1$ and for all $n > 0$, we have that

$$[\widetilde{\gamma}_n N, \widetilde{\gamma}_m N]^{\sim} \leq \widetilde{\gamma}_{n+m} N.$$

Let k be an arbitrary positive natural number. We want to show that

$$[\widetilde{\gamma}_k N, \widetilde{\gamma}_{m+1} N]^{\sim} \leq \widetilde{\gamma}_{k+m+1} N.$$

We have that

$$\left[[\widetilde{\gamma}_k N, N]^{\sim}, \widetilde{\gamma}_m N\right]^{\sim} = [\widetilde{\gamma}_{k+1} N, \widetilde{\gamma}_m N]^{\sim} \overset{\text{hyp}}{\leq} \widetilde{\gamma}_{k+m+1} N$$

and

$$\left[[\widetilde{\gamma}_k N, \widetilde{\gamma}_m N]^{\sim}, N\right]^{\sim} \underset{6.5(1)}{\overset{\text{hyp}}{\leq}} [\widetilde{\gamma}_{k+m} N, N]^{\sim} \leq \widetilde{\gamma}_{k+m+1} N.$$

As $k+m \geq 2$, we have that the group $\widetilde{\gamma}_{k+m+1}N$ is an intersection of normal definable subgroups of G. Thus by the three subgroups lemma (Corollary 6.11) we have that

$$\left[\widetilde{\gamma}_k N, \widetilde{\gamma}_{m+1} N\right]^{\sim} = \left[[\widetilde{\gamma}_m N, N]^{\sim}, \widetilde{\gamma}_k N\right]^{\sim} \leq \widetilde{\gamma}_{k+m+1} N$$

and the lemma is established. □

Lemma 6.16. *Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then, for any natural numbers $n \geq 2$, i and j we have that*

$$\left[\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\right]^{\sim} \leq \widetilde{\gamma}_{n+i} N$$

where $\left[\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\right]^{\sim}$ for $i = j = 0$ equals $\widetilde{\gamma}_n N$.

Proof. Note first that as n is at least 2, the group $\widetilde{\gamma}_n N$ is an intersection of normal A-definable groups. Thus for i equal to 0, we have that $\tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_j G\tilde{]} \leq \widetilde{\gamma}_n N$ by Lemma 6.5(2).

Now, let i be equal to 1. Note first that by Lemma 6.5(1) +(2),

$$\tilde{\Big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{1+j}(N, G^j)\tilde{\Big]} \leq \tilde{\Big[}\widetilde{\gamma}_n N, \tilde{[}N, G\tilde{]}\tilde{\Big]}. \quad (*)$$

Furthermore, we have the following:

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_n N, G\tilde{]}, N\tilde{\Big]} \overset{6.5(1)+(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, N\tilde{]} = \widetilde{\gamma}_{n+1} N,$$

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_n N, N\tilde{]}, G\tilde{\Big]} \overset{6.5(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, N\tilde{]} = \widetilde{\gamma}_{n+1} N.$$

Hence, as $\widetilde{\gamma}_{n+1} N$ is the intersection of A-definable subgroups, the three subgroups lemma (Corollary 6.11) yields that $\tilde{[}\widetilde{\gamma}_n N, \tilde{[}N, G\tilde{]}\tilde{]}$ is contained in $\widetilde{\gamma}_{n+1} N$. Now, by $(*)$ we conclude for i equals to 1.

If i is greater than 1, we have that

$$\tilde{\Big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\tilde{\Big]} \overset{6.5(1)+(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_i N\tilde{]}.$$

By Lemma 6.15, we obtain that $\tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_i N\tilde{]}$ is contained in $\widetilde{\gamma}_{n+i} N$ which finishes the proof. □

The following lemma is [21, Lemma 7] generalized to our framework.

Lemma 6.17. *Let N be a A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G and suppose that there exists a natural number $m > 0$ such that $\widetilde{\gamma}_{m+1}(N, G^m) \lesssim \tilde{[}N, N\tilde{]}$. Then, for all natural numbers $r > 0$ we have that*

$$\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1}) \leq \widetilde{\gamma}_{r+1} N.$$

Proof. We start this proof with the following claim.

Claim. *Let X be an ind-definable normal subgroup of G. Then for any $n > 0$, we have that*

$$\widetilde{\gamma}_{n+2}(X, N, G^n) \leq \prod_{i=0}^{n} \tilde{\Big[}\widetilde{\gamma}_{i+1}(X, G^i), \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{\Big]}. \tag{6.1}$$

Proof of the claim. We prove the claim by induction on $n > 0$. Let n be equal to 1. Trivially we have that

$$\tilde{[}N, G, X\tilde{]} \leq \tilde{\Big[}X, \tilde{[}N, G\tilde{]}\tilde{\Big]} \cdot \tilde{\Big[}\tilde{[}X, G\tilde{]}, N\tilde{\Big]}$$

and

$$\tilde{[}G, X, N\tilde{]} \leq \tilde{\Big[}X, \tilde{[}N, G\tilde{]}\tilde{\Big]} \cdot \tilde{\Big[}\tilde{[}X, G\tilde{]}, N\tilde{\Big]}.$$

The three subgroups lemma (Corollary 6.11) insures that

$$\tilde{[}X, N, G\tilde{]} \leq \tilde{\Big[}X, \tilde{[}N, G\tilde{]}\tilde{\Big]} \cdot \tilde{\Big[}\tilde{[}X, G\tilde{]}, N\tilde{\Big]}$$

and so the claim holds for $n = 1$.

Now, assume the claim holds for some $n > 0$. We compute:

$$\begin{array}{rcl}\widetilde{\gamma}_{n+3}(X,N,G^{n+1}) & = & \tilde{\Big[}\widetilde{\gamma}_{n+2}(X,N,G^n),G\tilde{\Big]} \\ & \overset{IH}{\underset{6.5(1)}{\leq}} & \tilde{\Big[}\prod_{i=0}^{n}\tilde{\big[}\widetilde{\gamma}_{i+1}(X,G^i),\, \widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\big]},G\tilde{\Big]}.\end{array}$$

As all factors are type-definable normal subgroups of G we may apply Lemma 6.7 finitely many times to the last expression and continue the computation:

$$\leq \prod_{i=0}^{n}\tilde{\Big[}\tilde{\big[}\widetilde{\gamma}_{i+1}(X,G^i),\, \widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\big]},G\tilde{\Big]}. \tag{6.2}$$

To simplify notation, we let $X_i = \widetilde{\gamma}_{i+1}(X,G^i)$ and $N_j = \widetilde{\gamma}_{j+1}(N,G^j)$. Now, fix some i less or equal to n. We obtain that

$$\begin{array}{rcl}\tilde{\Big[}\tilde{\big[}\widetilde{\gamma}_{i+1}(X,G^i),G\tilde{\big]},\widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\Big]} & = & \tilde{\big[}\widetilde{\gamma}_{i+2}(X,G^{i+1}),\widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\big]} \\ & = & \tilde{[}X_{i+1},N_{n-i}\tilde{]}\end{array}$$

and

$$\begin{array}{rcl}\tilde{\Big[}\tilde{\big[}\widetilde{\gamma}_{n-i+1}(N,G^{n-i}),G\tilde{\big]},\widetilde{\gamma}_{i+1}(X,G^i)\tilde{\Big]} & = & \tilde{\big[}\widetilde{\gamma}_{n-i+2}(N,G^{n+1-i}),\widetilde{\gamma}_{i+1}(X,G^i)\tilde{\big]} \\ & = & \tilde{[}N_{n-i+1},X_i\tilde{]} \\ & = & \tilde{[}X_i,N_{n-i+1}\tilde{]}.\end{array}$$

As the groups on the right are intersections of definable subgroups of G, using the approximate three subgroups lemma (Corollary 6.11), we obtain the following inequation for the ith factor of (6.2):

$$\tilde{\Big[}\tilde{\big[}\widetilde{\gamma}_{i+1}(X,G^i),\, \widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\big]},G\tilde{\Big]} \leq \tilde{[}X_{i+1},N_{n-i}\tilde{]}\cdot\tilde{[}X_i,N_{n-i+1}\tilde{]}.$$

Over all, we get that

$$\widetilde{\gamma}_{n+3}(X,N,G^n) \leq \prod_{i=0}^{n+1}\tilde{[}X_i,N_{n-i+1}\tilde{]} = \prod_{i=0}^{n+1}\tilde{\big[}\widetilde{\gamma}_{i+1}(X,G^i),\, \widetilde{\gamma}_{n+1-i+1}(N,G^{n+1-i})\tilde{\big]}.$$

$\square_{\text{claim}}$

Now, we prove the Lemma by induction on $r > 0$. By Corollary 6.4, the almost inequality $\widetilde{\gamma}_{m+1}(N,G^m) \lesssim \tilde{[}N,N\tilde{]}$ implies immediately $\widetilde{\gamma}_{m+1}(N,G^m) \leq \tilde{[}N,N\tilde{]}$. Thus, for r equals to 1 the lemma holds trivially by the hypothesis. Assume that the result holds for a given r greater or equal to 1. We want to prove that

$$\widetilde{\gamma}_{(r+1)m+1}(N^{r+1},G^{(r+1)m-r}) \leq \widetilde{\gamma}_{r+2}N.$$

Now consider equation (6.1) with $n = (r+1)m-r$ and X replaced by $\widetilde{\gamma}_r N^r$. This gives us:

$$\begin{array}{rcll}\widetilde{\gamma}_{(r+1)m+1}(N^{r+1},G^{(r+1)m-r}) & = & \widetilde{\gamma}_{((r+1)m-r)+2}(\widetilde{\gamma}_r N,N,G^{(r+1)m-r}) & (6.3) \\ & \leq & \displaystyle\prod_{i=0}^{(r+1)m-r}\tilde{\big[}\widetilde{\gamma}_{i+1}(\widetilde{\gamma}_r N,G^i),\widetilde{\gamma}_{n-i+1}(N,G^{n-i})\tilde{\big]}. & (6.4)\end{array}$$

The group on the left hand side is the one we want to analyze. The goal is to prove that all factors on the right hand side are contained in $\widetilde{\gamma}_{r+2}N$. So, we consider the factor indexed by i.

Supoose first that i is greater than $rm-r$. By induction hypothesis, we have that

$$\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1}) \leq \widetilde{\gamma}_{r+1}N.$$

As $\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1})$ is normal in G and an intersection of A-definable groups, using Lemma 6.5 (2) we obtain that $\widetilde{\gamma}_{r+i}(N^r, G^i) \leq \widetilde{\gamma}_{r+1}N$ and

$$\begin{aligned}\tilde{\left[\widetilde{\gamma}_{i+1}(\widetilde{\gamma}_r N, G^i), \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\right]} &\overset{6.5(1)}{\leq} \tilde{\left[\widetilde{\gamma}_{r+1}N, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\right]} \\ &\overset{6.16}{\leq} \widetilde{\gamma}_{r+2}N.\end{aligned}$$

Now, assume that $i \leq rm-r$. By the case $r=1$, we have that $\widetilde{\gamma}_{m+1}(N, G^m) \leq \tilde{[N,N]}$. As $n-i$ is greater than or equal to m and $\widetilde{\gamma}_{m+1}(N, G^m)$ is an intersection of normal subgroups of G, we also have that $\widetilde{\gamma}_{n-i+1}(N, G^{n-i}) \leq \tilde{[N,N]}$. So we may compute:

$$\begin{aligned}\tilde{\left[\widetilde{\gamma}_{i+1}((\widetilde{\gamma}_r N^r), G^i), \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\right]} &\overset{6.5(1)}{\leq} \tilde{\left[\widetilde{\gamma}_{i+r}(N^r, G^i), \tilde{[N,N]}\right]} \\ &\overset{6.16}{\leq} \widetilde{\gamma}_{r+2}N.\end{aligned}$$

Hence all factors, and therefore $\widetilde{\gamma}_{(r+1)m+1}(N^{r+1}, G^{(r+1)m-r})$, are contained in $\widetilde{\gamma}_{r+2}N$. This finishes the proof. $\square$

Now, we are ready to generalize Hall's nilpotency criteria (Fact 6.10) to $\widetilde{\mathfrak{M}}_c$-groups.

Corollary 6.18. *Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. If N is almost nilpotent of class m and $G/\tilde{[N,N]}$ is almost nilpotent of class n then G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$.*

Remark 6.19. Note that for an almost nilpotent subgroup H of an $\widetilde{\mathfrak{M}}_c$ group, if $\widetilde{\gamma}_{n+1}H$ is trivial, then H is almost nilpotent of class at most $n+1$ by Corollary 6.14, whereas for nilpotent groups triviality of $\widetilde{\gamma}_{n+1}H$ yields that H is nilpotent of class at most n. This explains the extra plus 1 in the previous Corollary in comparison with the original Hall's nilpotency criteria.

Proof. By hypothesis and Lemma 6.14 we have that

$$\widetilde{\gamma}_{m+1}N = 1 \quad \text{and} \quad \widetilde{\gamma}_{n+1}G \leq \tilde{[N,N]}. \quad (*)$$

Hence

$$\widetilde{\gamma}_{n+1}(N, G^n) \leq \tilde{[N,N]}$$

and whence N satisfies the hypothesis of Lemma 6.17. Thus

$$\widetilde{\gamma}_{rn+1}(N^r, G^{rn-r+1}) \leq \widetilde{\gamma}_{r+1}N \tag{6.5}$$

holds for all natural numbers r.

Claim. *Let $f(x) = \binom{x+1}{2}n - \binom{x}{2}$. For every i greater than 1, we obtain that*

$$\widetilde{\gamma}_{f(i)+1}G \leq \widetilde{\gamma}_{i+1}N.$$

Proof of the claim. We prove the claim by induction on $i \geq 2$.

So let i be equal to 2. We compute:

$$\widetilde{\gamma}_{f(2)+1}G = \widetilde{\gamma}_{3n}G = \widetilde{\gamma}_{2n}(\widetilde{\gamma}_{n+1}G, G^{2n-1}) \underset{6.5(1)}{\overset{(*)}{\leq}} \widetilde{\gamma}_{2n}\left(\tilde{[}N,N\tilde{]}, G^{2n-1}\right) \overset{(6.5)}{\leq} \widetilde{\gamma}_3 N.$$

Now, suppose the claim holds for $i \geq 2$. We show that the claim holds for $i+1$:

$$\begin{aligned} \widetilde{\gamma}_{f(i+1)+1}G &= \widetilde{\gamma}_{(i+1)n-i+1}\left(\widetilde{\gamma}_{f(i)+1}G, G^{(i+1)n-i}\right) \\ &\underset{6.5(1)}{\overset{\text{hyp}}{\leq}} \widetilde{\gamma}_{(i+1)n-i+1}\left(\widetilde{\gamma}_{i+1}N, G^{(i+1)n-(i+1)+1}\right) \\ &\underset{6.5(1)}{\overset{\text{hyp}}{\leq}} \widetilde{\gamma}_{(i+1)n+1}\left(N^{i+1}, G^{(i+1)n-(i+1)+1}\right) \\ &\overset{6.5}{\leq} \widetilde{\gamma}_{i+2}N. \end{aligned}$$

This finishes the induction and the proof of the claim. $\square_{\text{claim}}$

Choosing i to be m we get that

$$\widetilde{\gamma}_{f(m)+1}G \leq \widetilde{\gamma}_{m+1}N = \{1\}.$$

So Lemma 6.14 yields that G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$. $\square$

Corollary 6.20. *Let H and K be A-ind-definable normal subgroups of an $\widetilde{\mathfrak{M}}_c$-group G.*

1. *If $\tilde{[}H,H\tilde{]} = \tilde{[}G,G\tilde{]}$, then for all $r \geq 2$, we have $\widetilde{\gamma}_r H = \widetilde{\gamma}_r G$.*
2. *If $\tilde{[}H,K\tilde{]}$ and $\tilde{[}H,H\tilde{]}$ are contained in $\tilde{[}K,K\tilde{]}$, then for all $r \geq 2$, the almost commutator $\widetilde{\gamma}_r H$ is contained in $\widetilde{\gamma}_r K$.*

Proof. 1. As H is a subgroup of G, we have that $\widetilde{\gamma}_r H \leq \widetilde{\gamma}_r G$ holds trivially for all $r \geq 2$. We prove the inverse inclusion by induction on r. For r equals to 2, the statement holds by hypothesis. Now suppose that the statement holds for all natural numbers smaller than $r > 2$. Thus,

$$\widetilde{\gamma}_r G \leq \widetilde{\gamma}_r(H^{r-1}, G).$$

Furthermore, $\tilde{[}H,G\tilde{]} \leq \tilde{[}H,H\tilde{]}$, hence we may apply Lemma 6.17 with $m = 1$ and obtain that

$$\widetilde{\gamma}_r(H^{r-1}, G) \leq \widetilde{\gamma}_r H$$

which finishes the proof.

2. Consider $L = HK$. Then we can compute that

$$\tilde{[}L,L\tilde{]} = \tilde{[}HK,HK\tilde{]} \overset{6.7}{\leq} \tilde{[}H,H\tilde{]} \cdot \tilde{[}K,K\tilde{]} \cdot \tilde{[}H,K\tilde{]} = \tilde{[}K,K\tilde{]}.$$

By the first part of the corollary we can conclude that $\widetilde{\gamma}_r H^r \leq \widetilde{\gamma}_r L^r = \widetilde{\gamma}_r K^r$.

$\square$

Other applications of the almost three subgroups lemma and results on almost nilpotent groups

Using symmetry of the almost centralizer, the three subgroups lemma and the definabilily of the almost centralizer, we may generalize a theorem due to Hall [31, Satz III.2.8] for the ordinary centralizer to our context.

Proposition 6.21. *Let G be an $\widetilde{\mathfrak{M}}_c$-group, $N_0 \geq N_1 \geq \cdots \geq N_m \geq \ldots$ be a descending sequence of A-definable normal subgroups of G, and H be an A-ind-definable normal subgroup of G. Suppose that for all $i \in \omega$, we have $H \lesssim \widetilde{C}_G(N_i/N_{i+1})$. We define for $i > 0$,*

$$H_i := \bigcap_{k \in \omega} \widetilde{C}_H(N_k/N_{k+i}).$$

Then we have that for all positive natural numbers i and j, the group H_i is almost contained in $\widetilde{C}_G(H_j/H_{i+j})$, the group H is almost contained in $\widetilde{C}^i_G\left(H \middle/ \widetilde{C}_G(N_{j-1}/N_{i+j})\right)$ and therefore $[\widetilde{\gamma}_{i+1}H, N_{j-1}\tilde{]} \leq N_{i+j}$.

Remark 6.22. The non-approximate version [31, Satz III.2.8] states that for H_i defined as $\bigcap_{k<\omega} C_H(N_k/N_{k+i})$ we have that for all positive natural numbers i and j, $[H_i, H_j] \leq H_{i+j}$ and $[\gamma_{i+1}H, N_{j-1}] \leq N_{i+j}$.

Proof. Note that H_i is equal to $\bigcap_{k\in\omega} \widetilde{C}_G(N_k/N_{k+i}) \cap H$ and thus the intersection of an ind-definable subgroup and boundedly many definable subgroups. So H_i is as well an ind-definable subgroup of G.

As $\widetilde{C}_G(N_k/N_{k+i+j})$ is definable for any natural number k, Properties 3.8 (9) yields that

$$H_i \lesssim \widetilde{C}_G(H_j/H_{i+j}) = \widetilde{C}_G\left(H_j \middle/ \bigcap_{k<\omega} \widetilde{C}_G(N_k/N_{k+i+j})\right)$$

if and only if for all natural number k we have that

$$H_i \lesssim \widetilde{C}_G\left(H_j \middle/ \widetilde{C}_G(N_k/N_{k+i+j})\right).$$

So it is enough to show the latter result for any natural number $k \in \omega$. So fix some k, i and j in ω. By the definition of H_j we have that $H_j \leq \widetilde{C}_G(N_{k+i}/N_{k+i+j})$. Symmetry modulo definable subgroups for almost centralizers yields that $N_{k+i} \lesssim \widetilde{C}_G(H_j/N_{k+i+j})$. This implies that

$$H_i \quad \leq \quad \widetilde{C}_G(N_k/N_{k+i}) \leq \widetilde{C}_G\left(N_k \middle/ \widetilde{C}_G(H_j/N_{k+i+j})\right). \tag{6.6}$$

Exchanging the role of i and j we obtain as well that

$$H_j \quad \leq \quad \widetilde{C}_G\left(N_k \middle/ \widetilde{C}_G(H_i/N_{k+j+i})\right) = \widetilde{C}_G\left(N_k \middle/ \widetilde{C}_G(H_i/N_{k+i+j})\right). \tag{6.7}$$

Using again symmetry modulo definable subgroups for almost centralizers to (6.6), we get:

$$N_k \quad \lesssim \quad \widetilde{C}_G\left(H_i \middle/ \widetilde{C}_G(H_j/N_{k+i+j})\right). \tag{6.8}$$

Working in G/N_{k+i+j}, we can apply the three subgroups lemma (Theorem 3.19) to the equalities (6.7) and (6.8) since all N_i's and all H_i's normalize each other and obtain

$$H_i \lesssim \widetilde{C}_G\left(H_j \middle/ \widetilde{C}_G(N_k/N_{k+i+j})\right).$$

As k was arbitrary, this establishes the first part of the theorem.

In particular, we have that for any natural numbers i and j greater than 0

$$\begin{aligned} H_1 &\lesssim \widetilde{C}_G(H_1/H_2) \lesssim \widetilde{C}_G\left(H_1 \middle/ \widetilde{C}_G(H_1/H_3)\right) = \widetilde{C}_G^2(H_1/H_3) \\ &\lesssim \cdots \lesssim \widetilde{C}_G^i(H_1/H_{i+1}) \lesssim \widetilde{C}_G^i\left(H_1 \middle/ \widetilde{C}_G(N_{j-1}/N_{i+j})\right) \end{aligned}$$

By hypothesis we have that H_1 is a bounded intersection of groups which are commensurate with H and whence it is itself commensurate with H. As two commensurate groups have the same almost centralizer, the same almost inclusion holds for H which finishes the proof. □

Using the previous result and definability of the almost centralizers, we may find a version of [7, Lemma 2.4] in terms of the almost centralizer:

Corollary 6.23. *Let H be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then for any $0 < i < j$, we have that*

$$H \lesssim \widetilde{C}_G^i\left(H \middle/ \widetilde{C}_G\left(\widetilde{C}_G^j(H) \middle/ \widetilde{C}_G^{j-i-1}(H)\right)\right)$$

Proof. For $k < 2j-1$, we let $N_k = \widetilde{C}_G^{2j-1-k}(H)$ and for $k \geq 2j-1$, we let N_k be the trivial group. As G is an $\widetilde{\mathfrak{M}}_c$-group, all N_k are definable. Note that for any natural number n, the almost centralizer $\widetilde{C}_G^n(H)$ is definable and $\widetilde{C}_G^{n+1}(H) = \widetilde{C}_G(H/\widetilde{C}_G^n(H))$ is contained in itself. Hence, symmetry of the almost centralizer (Theorem 3.13) yield that

$$H \lesssim \widetilde{C}_G\left(\widetilde{C}_G^{n+1}(H) \middle/ \widetilde{C}_G^n(H)\right),$$

whence

$$H \lesssim \widetilde{C}_G(N_k/N_{k+1}).$$

So we may apply Proposition 6.21 to the ind-definable subgroup H and the sequence of definable groups N_i. This gives us that

$$H \lesssim \widetilde{C}_G^i\left(H \middle/ \widetilde{C}_G\left(N_{j-1}/N_{i+j}\right)\right) = \widetilde{C}_G^i\left(H \middle/ \widetilde{C}_G\left(\widetilde{C}_G^j(H) \middle/ \widetilde{C}_G^{j-i-1}(H)\right)\right)$$

□

Using the new notion of almost commutator, we may state the previous lemma in this terminology which resembles more to the ordinary result.

Corollary 6.24. *Let H be an A-ind-definable normal subgroup of the $\widetilde{\mathfrak{M}}_c$-group G. Then for any $0 < i < j$, we have that*

$$[\widetilde{\gamma}_{i+1} H, \widetilde{C}_G^j(H)\tilde{]} \leq \widetilde{C}_G^{j-i-1}(H).$$

Proof. We have that

$$H \lesssim \widetilde{C}^i_G\Big(H\Big/\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)\Big) = \widetilde{C}_G\Big(H\Big/\widetilde{C}^{i-1}_G\Big(H\Big/\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)\Big)\Big)$$

Using that the iterated almost centralizer of an ind-definable subgroup of an $\widetilde{\mathfrak{M}}_c$-group is definable as well as that an ind-definable subgroup modulo a definable subgroup remains ind-definable, we have that $\widetilde{C}^\ell_G\Big(H\Big/\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)\Big)$ is definable for any natural number ℓ. Hence, the above yields that

$$\widetilde{[}H,H\widetilde{]} \leq \widetilde{C}^{i-1}_G\Big(H\Big/\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)\Big).$$

Iterating this process gives us

$$\widetilde{\gamma}_i H \leq \widetilde{C}_G\Big(H\Big/\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)\Big).$$

As the almost centralizer $\widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big)$ is again definable, we get

$$\widetilde{\gamma}_{i+1} H \leq \widetilde{C}_G\Big(\widetilde{C}^j_G(H)\Big/\widetilde{C}^{j-i-1}_G(H)\Big).$$

By the same argument, we obtain the final inequation:

$$\widetilde{[}\widetilde{\gamma}_{i+1} H, \widetilde{C}^j_G(H)\widetilde{]} \leq \widetilde{C}^{j-i-1}_G(H).$$

□

In the next lemma, we use the almost three subgroups lemma in terms of the almost commutator to generalize [7, Lemma 2.5] to our framework.

Lemma 6.25. *Let H and K be two A-ind-definable normal subgroups of G with $K \leq H$ and $\ell > 0$. If*

$$\widetilde{C}_G(\widetilde{\gamma}_t K) \sim \widetilde{C}_G(\widetilde{\gamma}_t H) \quad t = 1,\ldots,\ell$$

then $\widetilde{C}^\ell_G(K) \sim \widetilde{C}^\ell_G(H)$.

Proof. The case ℓ equals 1 is trivial. So let's assume that the lemma holds for $\ell - 1$. We need to prove the following intermediate result:

Claim. $\widetilde{[}\widetilde{\gamma}_{\ell-t} H, \widetilde{C}^\ell_G(K)\widetilde{]} \leq \widetilde{C}^t_G(H)$ *holds for all $t = 0,\ldots,\ell-1$.*

Proof. We show the claim by induction on the tuple (ℓ,t) (ordered lexicographically) with $t < \ell$. First we treat the cases $(\ell,0)$ for any positive natural number ℓ:
Replacing H by K, i by $\ell-1$, and j by ℓ in Corollary 6.24, we obtain $\widetilde{[}\widetilde{\gamma}_\ell K, \widetilde{C}^\ell_G(K)\widetilde{]} = 1$. This implies that $\widetilde{C}^\ell_G(K)$ is almost contained in $\widetilde{C}_G(\widetilde{\gamma}_\ell K)$ which is, by the hypothesis of the lemma, commensurate with $\widetilde{C}_G(\widetilde{\gamma}_\ell H)$. Thus $\widetilde{C}^\ell_G(K) \lesssim \widetilde{C}_G(\widetilde{\gamma}_\ell H)$ or in other words $\widetilde{[}\widetilde{\gamma}_\ell H, \widetilde{C}^\ell_G(K)\widetilde{]} = 1$. Hence the claim holds for $(\ell,0)$ with $\ell > 0$.

Now, let $0 < t < \ell$ and assume additionally that the claim holds for any tuple $(k,s) < (\ell,t)$ in the lexicographical order.
Then using Lemma 6.5 (1) and the induction hypothesis for $(\ell,t-1)$ (in the equation

marked as $(*)$ below) and for $(\ell-1, t-1)$ (in the equation marked as $(**)$ below) we may compute

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, K\tilde{]}, \widetilde{C}_G^{\ell}(K)\tilde{\Big]}\tilde{]} \underset{K\leq H}{\overset{6.5(1)}{\leq}} \tilde{\Big[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, H\tilde{]}, \widetilde{C}_G^{\ell}(K)\tilde{\Big]} = \tilde{[}\widetilde{\gamma}_{\ell-(t-1)}H, \widetilde{C}_G^{\ell}(K)\tilde{]} \overset{(*)}{\leq} \widetilde{C}_G^{t-1}(H)$$

and

$$\tilde{\Big[}\widetilde{\gamma}_{\ell-t}H, \tilde{[}K, \widetilde{C}_G^{\ell}(K)\tilde{]}\tilde{\Big]} \leq \tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell-1}(K)\tilde{]} = \tilde{[}\widetilde{\gamma}_{(\ell-1)-(t-1)}H, \widetilde{C}_G^{\ell-1}(H)\tilde{]} \overset{(**)}{\leq} \widetilde{C}_G^{t-1}(H).$$

Thus by Corollary 6.11 we have

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_{l-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{\Big]} \leq \widetilde{C}_G^{t-1}(H).$$

As $t-1$ is less than ℓ, we have, by the hypothesis of the outer induction, that $\widetilde{C}_G^{t-1}(H)$ is commensurate with $\widetilde{C}_G^{t-1}(K)$ and so $\tilde{\Big[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{\Big]}$ is almost contained in $\widetilde{C}_G^{t-1}(K)$. As $\widetilde{C}_G^{t-1}(K)$ is A-definable, using Corollary 6.4, we obtain that

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{\Big]} \leq \widetilde{C}_G^{t-1}(K).$$

Thus $\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}$ is almost contained in $\widetilde{C}_G^{t}(K)$ which is commensurate once more with $\widetilde{C}_G^{t}(H)$ by the outer induction hypothesis. Again by Corollary 6.4 almost contained can be replaced by contained, which gives us

$$\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]} \leq \widetilde{C}_G^{t}(H).$$

Thus the claim holds for the tuple (ℓ, t) which finishes the induction and hence the proof of the claim. $\square_{\text{(claim)}}$

Now taking t equals to $\ell-1$, we obtain $\tilde{[}H, \widetilde{C}_G^{\ell}(K)\tilde{]} \leq \widetilde{C}_G^{\ell-1}(H)$ which implies that $\widetilde{C}_G^{\ell}(K)$ is almost contained in $\widetilde{C}_G^{\ell}(H)$. On the other hand, we have that

$$\tilde{[}K, \widetilde{C}_G^{\ell}(H)\tilde{]} \underset{K\leq H}{\overset{6.5(1)}{\leq}} \tilde{[}H, \widetilde{C}_G^{\ell}(H)\tilde{]} \overset{6.9}{\leq} \widetilde{C}_G^{\ell-1}(H) \overset{\text{hyp.}}{\sim} \widetilde{C}_G^{\ell-1}(K).$$

Again by Corollary 6.3 we obtain that $\tilde{[}K, \widetilde{C}_G^{\ell}(H)\tilde{]} \leq \widetilde{C}_G^{\ell-1}(K)$ and so $\widetilde{C}_G^{\ell}(H)$ is almost contained in $\widetilde{C}_G^{\ell}(K)$. Combining these two results, we obtain that $\widetilde{C}_G^{\ell}(K)$ is commensurate with $\widetilde{C}_G^{\ell}(H)$ which finishes the proof. $\square$

We finish this section with another result on almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups which does not use the almost three subgroups lemma.

Lemma 6.26. *Let G be almost nilpotent $\widetilde{\mathfrak{M}}_c$-group and N be a nontrivial intersection of A-definable normal subgroups of G. Then $\tilde{[}N, G\tilde{]}$ is properly contained in N and $N \cap \widetilde{Z}(G)$ is a nontrivial subgroup of G. In particular, any minimal A-invariant normal subgroup of G is contained in the almost center of G.*

Proof. As N is an intersection of A-definable normal subgroups of G and we have trivially that $N \lesssim \widetilde{C}_G(G/N)$, the group $\tilde{[}N, G\tilde{]}$ is contained in N. Additionally, the commutator $\tilde{[}N, G\tilde{]}$ is also contained in $\tilde{[}G, G\tilde{]}$ by Lemma 6.5. Inductively we obtain $\widetilde{\gamma}_{i+1}(N, G^i) \leq$

$N \cap \widetilde{\gamma}_{i+1}G$. As G is almost nilpotent $\widetilde{\gamma}_m G$ is trivial for some natural number m. Hence $\tilde{[}N,G\tilde{]}$ has to be properly contained in N because if not $\widetilde{\gamma}_m(N,G^{m-1})$ would be equal to N as well. This proves the first part of the Lemma.

Moreover, again by Lemma 6.5, we have that $\widetilde{\gamma}_m(N,G^{m-1}) \leq \widetilde{\gamma}_m G$ and thus it is also trivial. Now choose n such that $\widetilde{\gamma}_{n+1}(N,G^n)$ is trivial and properly contained in $\widetilde{\gamma}_n(N,G^{n-1})$. Hence

$$\widetilde{\gamma}_n(N,G^{n-1}) \lesssim \widetilde{Z}(G).$$

Since the almost center of G is definable, Corollary 6.4 yields that $\widetilde{\gamma}_n(N,G^{n-1})$ is actually contained in $\widetilde{Z}(G)$. As additionally the group $\widetilde{\gamma}_n(N,G^{n-1})$ is nontrivial and contained in N, the subgroup $N \cap \widetilde{Z}(G)$ is nontrivial as well. $\square$

Part II

Corps

7 Fields in n-dependent theories

Macintyre showed in [44] that any ω-stable filed is algebraically closed. Cherlin and Shelah [8] generalized this result to superstable fields. However, less is known in the case of supersimple fields. Hrushovski proved that any infinite perfect bounded pseudo algebraically closed (PAC) field is supersimple [30] and conversely supersimple fields are perfect and bounded (Pillay and Poizat [51]), and it is conjectured that they are PAC. Another subject of interest is to analyze the number of Artin-Schreier extensions of certain fields. Using a suitable chain condition for uniformly definable subgroups, Kaplan, Scanlon and Wagner showed in [34] that infinite dependent fields are Artin-Schreier closed and simple fields have at most finitely many Artin-Schreier extensions. The latter result was generalized to NTP_2 fields by Chernikov, Kaplan and Simon [12].

We study groups and fields without the n-independence property. As pointed out in the preliminaries, the random n-hypergraph is n-dependent and simple but not dependent. One question we are interested in is the existence of a non combinatorial examples of n-dependent theories which have the independence property. And furthermore, which results of dependent theories can be generalized to n-dependent theories or more specifically which results of (super)stable theories remain true for (super)simple n-dependent theories? Beyarslan [4] constructed the random n-hypergraph in any pseudofinite field or, more generally, in any e-free perfect PAC field (PAC fields whose absolute Galois group is the profinite completion of the free group on e generators). Thus, those fields lie outside of the hierarchy of n-dependent fields.

In this chapter, we first give an example of a group with a simple 2-dependent theory which has the independence property. Additionally, in this group the A-connected component depends on the parameter set A. This establishes on the one hand a non combinatorial example of a proper 2-dependent theory and on the other hand shows that the existence of an absolute connected component in any dependent group cannot be generalized to 2-dependent groups. Using the Baldwin-Saxl condition for n-dependent groups (Proposition 2.6) and connectivity of a certain vector group established in Section 7.2 we deduce that n-dependent fields are Artin-Schreier closed (Section 7.3). Furthermore, we show in Section 7.4 that the theory of any non separably closed PAC field is not n-dependent for any natural numbers n, This was established by Duret for the case n equals to 1 [16]. In Section 7.5 we extend certain consequences which can be found in [34] for dependent valued fields with perfect residue field as well as in [32] by Jahnke and Koenigsmann for dependent henselian valued fields to the n-dependent context.

7.1 Example of a 2-dependent group

Let G be $\bigoplus_{\omega} \mathbb{F}_p$ where $\mathbb{F}_p$ is the finite field with p elements. We consider the structure $\mathfrak{M}$ defined as $(G, \mathbb{F}_p, 0, +, \cdot)$ where 0 is the neutral element, + is addition in G, and $\cdot$ is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to $\mathbb{F}_p$. This example in the case p equals 2 has been studied by Wagner in [63, Example 4.1.14]. He shows that it is simple and that the connected component G_A^0 for any parameter set A is equal to $\{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$. Hence, it is getting smaller and smaller while enlarging A, whence the absolute connected component, which exists in any dependent group, does not for this example.

Lemma 7.1. *The theory of $\mathfrak{M}$ eliminates quantifiers.*

Proof. Let $t_1(x;\bar{y})$ and $t_2(x;\bar{y})$ be two group terms in G and let ϵ be an element of $\mathbb{F}_p$. Observe that the atomic formula $t_1(x;\bar{y}) = t_2(x;\bar{y})$ (resp. $t_1(x;\bar{y}) \neq t_2(x;\bar{y})$) is equivalent to an atomic formula of the form $x = t(\bar{y})$ or $0 = t(\bar{y})$ (resp. $x \neq t(\bar{y})$ or $0 \neq t(\bar{y})$) for some group term $t(\bar{y})$. Note that $0 = t(\bar{y})$ as well as $0 \neq t(\bar{y})$ are both quantifier free formulas in the free variables $\bar{y}$. Furthermore, the atomic formulas $t_1(x;\bar{y}) \cdot t_2(x;\bar{y}) = \epsilon$ and $t_1(x;\bar{y}) \cdot t_2(x;\bar{y}) \neq \epsilon$ are equivalent to a boolean combination of atomic formulas of the form $x \cdot x = \epsilon_x$, $x \cdot t_i(\bar{y}) = \epsilon_i$ and $t_j(\bar{y}) \cdot t_k(\bar{y}) = \epsilon_{jk}$ (a quantifier free formula in the free variables $\bar{y}$) with $t_i(\bar{y})$ group terms and ϵ_x, ϵ_i, and ϵ_{jk} elements of $\mathbb{F}_p$. Thus, a quantifier free formula $\varphi(x,\bar{y})$ is equivalent to a finite disjunction of formulas of the form

$$\phi(x;\bar{y}) = \psi(\bar{y}) \wedge x \cdot x = \epsilon \wedge \bigwedge_{i \in I_0} x = t_i^0(\bar{y}) \wedge \bigwedge_{i \in I_1} x \neq t_i^1(\bar{y}) \wedge \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$$

where $t_i^j(\bar{y})$ are group terms, ϵ, ϵ_i are elements of $\mathbb{F}_p$, and $\psi(\bar{y})$ is a quantifier free formula in the free variables $\bar{y}$. If I_0 is nonempty, the formula $\exists x \phi(x,\bar{y})$ is equivalent to

$$\psi(\bar{y}) \wedge \bigwedge_{j,\ell \in I_0} t_j^0(\bar{y}) = t_\ell^0(\bar{y}) \wedge t_i^0(\bar{y}) \cdot t_i^0(\bar{y}) = \epsilon \wedge \bigwedge_{j \in I_1} t_i^0(\bar{y}) \neq t_j^1(\bar{y}) \wedge \bigwedge_{j \in I_2} t_i^0(\bar{y}) \cdot t_j^2(\bar{y}) = \epsilon_j$$

for any $i \in I_0$. Now, we assume that I_0 is the empty set. If there exists an element x' such that $x' \cdot z_i = \epsilon_i$ for given $z_0, \dots, z_m$ in G and $\epsilon_i \in \mathbb{F}_p$, one can always find an element x such that $x \cdot x = \epsilon$ and $x \neq v_j$ for given $v_0, \dots, v_q$ in G which still satisfies $x \cdot z_i = \epsilon_i$ by modifying x' at some large enough coordinate. Hence, it is enough to find a quantifier free condition which is equivalent to $\exists x \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$.

For $i \in \mathbb{F}_p$, let

$$Y_i = \{j \in I_2 : \epsilon_j = i\}.$$

Then $\exists x \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$ is equivalent to

$$\bigwedge_{i=0}^{p-1} \bigwedge_{j \in Y_i} t_j^2(\bar{y}) \notin \left\{ \sum_{k \in Y_0} \lambda_k^0 t_k^2(\bar{y}) + \cdots + \sum_{k \in Y_i \setminus j} \lambda_k^i t_k^2(\bar{y}) : \lambda_k^\ell \in \mathbb{F}_p, \sum_{\ell=1}^{i} \sum_{k \in Y_\ell}^{k \neq j} \ell \cdot_{\mathbb{F}_p} \lambda_k^\ell \neq i \right\}$$

which finishes the proof. □

Lemma 7.2. *The structure $\mathfrak{M}$ is 2-dependent.*

Proof. We suppose, towards a contradiction, that $\mathfrak{M}$ has IP_2. By Fact 1.9 we can find a formula $\phi(\bar{y}_0, \bar{y}_1; x)$ with $|x| = 1$ which witnesses the 2-independence property. By the proof of Lemma 7.1 and as being 2-dependent is preserved under boolean combinations (Fact 1.10), it suffices to prove that none of the following formulas can witness the 2-independence property in the variables $(\bar{y}_0, \bar{y}_1; x)$:

- quantifier free formulas of the form $\psi(\bar{y}_0, \bar{y}_1)$,
- the formula $x \cdot x = \epsilon$ with ϵ in $\mathbb{F}_p$,
- formulas of the form $x = t(\bar{y}_0, \bar{y}_1)$ for some group term $t(\bar{y}_0, \bar{y}_1)$,
- formulas of the form $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ for some group term $t(\bar{y}_0, \bar{y}_1)$ and ϵ in $\mathbb{F}_p$.

As the atomic formula $\psi(\bar{y}_0, \bar{y}_1)$ does not depend on x and $x \cdot x = \epsilon$ does not depend on $\bar{y}_0$ nor $\bar{y}_1$ they cannot witness the 2-independence property in the variables $(\bar{y}_0, \bar{y}_1; x)$. Furthermore, as for given $\bar{a}$ and $\bar{b}$, the formula $x = t(\bar{a}, \bar{b})$ can be only satisfied by a single element, such a formula is also 2-dependent. Thus the only candidate left is a formula of the form $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ with $t(\bar{y}_0, \bar{y}_1)$ some group term in G and ϵ an element of $\mathbb{F}_p$. Thus, we suppose that the formula $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ has IP_2 and choose some elements $\{\bar{a}_i : i \in \omega\}$, $\{\bar{b}_i : i \in \omega\}$ and $\{c_I : I \subset \omega^2\}$ which witness it. As $t(\bar{y}_0, \bar{y}_1)$ is just a sum of elements of the tuples $\bar{y}_0$ and $\bar{y}_1$ and G is commutative, we may write this formula as $x \cdot (t_a(\bar{y}_0) + t_b(\bar{y}_1)) = \epsilon$ in which the term $t_a(\bar{y}_0)$ (resp. $t_b(\bar{y}_1)$) is a sum of elements of the tuple $\bar{y}_0$ (resp. $\bar{y}_1$). Let

$$S_{ij} := \{x : x \cdot (t_a(\bar{a}_i) + t_b(\bar{b}_j)) = \epsilon\}$$

be the set of realizations of the formula $x \cdot (t_a(\bar{a}_i) + t_b(\bar{b}_j)) = \epsilon$. Note, that an element c belongs to S_{ij} if and only if we have that $e_{ij}(c)$ defined as

$$e_{ij}(c) = c \cdot \Big(t_a(\bar{a}_i) + t_b(\bar{b}_j)\Big)$$

is equal to ϵ. Let i, l, j, and k be arbitrary natural numbers. Then,

$$\begin{aligned} e_{ij}(c) &= c \cdot \Big(t_a(\bar{a}_i) + t_b(\bar{b}_j)\Big) \\ &= c \cdot \Big((t_a(\bar{a}_i) + t_b(\bar{b}_k)) + (p-1)(t_a(\bar{a}_l) + t_b(\bar{b}_k)) + (t_a(\bar{a}_l) + t_b(\bar{b}_j))\Big) \\ &= e_{ik}(c) + (p-1)e_{lk}(c) + e_{lj}(c). \end{aligned}$$

If the element c belongs to $S_{ik} \cap S_{lk} \cap S_{lj}$, the terms $e_{ik}(c)$, $e_{lk}(c)$, and $e_{lj}(c)$ are all equal to ϵ. By the equality above we get that $e_{ij}(c)$ is also equal to ϵ and so c also belongs to S_{ij}.

Let $I = \{(1,1), (1,2), (2,2)\}$. Then $c_I \in S_{22} \cap S_{12} \cap S_{11}$ but $c_I \notin S_{21}$ which contradicts the previous paragraph letting i and k be equal to 2 and l and j be equal to 1. Thus the formula $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ is 2-dependent, hence all formulas in the theory of $\mathfrak{M}$ are 2-dependent and $\mathfrak{M}$ is 2-dependent. □

7.2 A special vector group

We first recall the definition of a vector group.

Definition 7.3. A *vector group* is a group isomorphic to a finite Cartesian power of the additive group of a field.

We need the following fact about vector groups.

Fact 7.4. *[34, Corollary 2.6] Let k be a perfect field, $n \in \omega$, and G be a closed connected 1-dimensional algebraic subgroup of $(k^{\mathrm{alg}},+)^n$ defined over k. Then G is isomorphic over k to $(k^{\mathrm{alg}},+)$.*

For the rest of the section, we fix an sufficiently saturated algebraically closed field $\mathbb{K}$ of characteristic $p > 0$ and we let $\wp(x)$ be the additive homomorphism $x \mapsto x^p - x$ on $\mathbb{K}$.

We analyze the following algebraic subgroups of $(\mathbb{K},+)^n$:

Definition 7.5. For a singleton a in $\mathbb{K}$, we let G_a be equal to $(\mathbb{K},+)$, and for a tuple $\bar{a} = (a_0,\ldots,a_{n-1}) \in \mathbb{K}^n$ with $n > 1$ we define:

$$G_{\bar{a}} = \{(x_0,\ldots,x_{n-1}) \in \mathbb{K}^n : \ a_0 \cdot \wp(x_0) = a_i \cdot \wp(x_i) \text{ for } 0 \leq i < n\}.$$

Recall that for an algebraic group G, we denote by G^0 the connected component of the unit element of G. Note that if G is definable over some parameter set A, its connected component G^0 coincides with the smallest A-definable algebraic subgroup of G of finite index. Our aim is to show that $G_{\bar{a}}$ is connected for certain choices of $\bar{a}$, namely $G_{\bar{a}}$ coincides with $G_{\bar{a}}^0$.

Lemma 7.6. *Let k be an algebraically closed subfield of $\mathbb{K}$, let G be a k-definable connected algebraic subgroup of $(\mathbb{K}^n,+)$ and let f be a k-definable homomorphism from G to $(\mathbb{K},+)$ such that for every $\bar{g} \in G$ there are polynomials $P_{\bar{g}}(X_0,\ldots,X_{n-1})$ and $Q_{\bar{g}}(X_0,\ldots,X_{n-1})$ in $k[X_0,\ldots,X_{n-1}]$ such that*

$$f(\bar{g}) = \frac{P_{\bar{g}}(\bar{g})}{Q_{\bar{g}}(\bar{g})}.$$

Then f is an additive polynomial in $k[X_0,\ldots,X_{n-1}]$. In fact, there exists natural numbers $m_0,\ldots,m_n$ such that f is of the form $\sum_{i=0}^{m_0} a_{i,0}X_0^{p^i} + \cdots + \sum_{i=0}^{m_n} a_{i,n}X_n^{p^i}$ with coefficients $a_{i,j}$ in k.

Proof. By compactness, one can find finitely many definable subsets D_i of G and polynomials $P_i(X_0,\ldots,X_{n-1})$ and $Q_i(X_0,\ldots,X_{n-1})$ in $k[X_0,\ldots,X_{n-1}]$ such that f is equal to $P_i(\bar{x})/Q_i(\bar{x})$ on D_i. Using [5, Lemma 3.8] we can extend f to a k-definable homomorphism $F : (\mathbb{K}^n,+) \to (\mathbb{K},+)$ which is also locally rational. Now, the functions

$$F_0(X) := F(X,0,\ldots,0), \ \ldots \ ,F_{n-1}(X) := F(0,\ldots,0,X)$$

are k-definable homomorphisms of $(\mathbb{K},+)$ to itself. Additionally, they are rational on a finite definable decomposition of $\mathbb{K}$, so they are rational on a cofinite subset of $\mathbb{K}$.

Hence every F_i is an additive polynomial in $k[X]$. As F itself was assumed to be a homomorphism from G to $(\mathbb{K},+)$, we obtain that

$$\begin{aligned} F(X_0,\dots,X_{n-1}) &= F(X_0,0,\dots,0)) + F(0,X_1,0,\dots,0))\cdots + F(0,\dots,0,X_{n-1}) \\ &= F_0(X_0)+\cdots+F_{n-1}(X_{n-1}) \end{aligned}$$

is an additive polynomial in $k[X_0,\dots,X_{n-1}]$ as it is a sum of additive polynomials. By [19, Proposition 1.1.5] it is of the desired form. □

Lemma 7.7. *Let $\bar{a}=(a_0,\dots,a_n)$ be a tuple in $\mathbb{K}^\times$. Then $G_{\bar{a}}$ is connected if and only if the set $\left\{\frac{1}{a_0},\dots,\frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent.*

Parts of the proof follows the one of [34, Lemma 2.8].

Proof. So suppose first that $\left\{\frac{1}{a_0},\dots,\frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-dependent. Thus we can find elements $b_0,\dots,b_{n-1}$ in $\mathbb{F}_p$ such that

$$b_0\cdot\frac{1}{a_0}+\cdots+b_{n-1}\frac{1}{a_{n-1}}=\frac{1}{a_n}.$$

Now, let $\bar{a}'$ be the tuple $\bar{a}$ restricted to its first n coordinates and fix some element $(x_0,\dots x_{n-1})$ in $G_{\bar{a}'}$. Let t be defined as $a_0(x_0^p-x_0)$. Hence, by the definition of $G_{\bar{a}'}$, we have that t is equal to $a_i(x_i^p-x_i)$ for any $i<n$. Furthermore, we have that $(x_0,\dots,x_{n-1},x)$ belongs to $G_{\bar{a}}$ if and only if

$$\begin{aligned} & t &&= a_n(x^p-x) \\ \Leftrightarrow\ & 0 &&= \frac{1}{a_n}t-(x^p-x) \\ \Leftrightarrow\ & 0 &&= \frac{b_0}{a_0}t+\cdots+\frac{b_{n-1}}{a_{n-1}}t-(x^p-x) \\ \Leftrightarrow\ & 0 &&= b_0\cdot(x_0^p-x_0)+\cdots+b_{n-1}\cdot(x_{n-1}^p-x_{n-1})-(x^p-x) \\ \Leftrightarrow\ & 0 &&= (b_0\cdot x_0+\cdots+b_{n-1}\cdot x_{n-1}-x)^p-(b_0\cdot x_0+\cdots+b_{n-1}x_{n-1}-x). \end{aligned}$$

In other words, $(x_0,\dots,x_{n-1},x)$ belongs to $G_{\bar{a}}$ if and only if $b_0\cdot x_0+\cdots+b_{n-1}x_{n-1}-x$ is an element of $\mathbb{F}_p$. With this formulation we consider the following subset of $G_{\bar{a}}$:

$$H=\{(x_0,\dots x_n)\in G_{\bar{a}}:(x_0,\dots x_{n-1})\in G_{\bar{a}'} \text{ and } b_0\cdot x_0+\dots b_{n-1}x_{n-1}-x_n=0\}$$

This is in fact a proper definable subgroup of $G_{\bar{a}}$ of finite index. Hence $G_{\bar{a}}$ is not connected.

We prove the other implication by induction on the length of the tuple $\bar{a}$ which we denote by n. Let $n=1$, then $G_{\bar{a}}$ is equal to $(\mathbb{K},+)$ and thus connected since the additive group of an algebraically closed field is always connected.

Let $\bar{a}=(a_0,\dots,a_n)$ be an $(n+1)$-tuple such that $\left\{\frac{1}{a_0},\dots,\frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent and suppose that the statement holds for tuples of length n. Define $\bar{a}'$ to be the restriction of $\bar{a}$ to the first n coordinates. Observe that the natural map $\pi:G_{\bar{a}}\to G_{\bar{a}'}$ is surjective since $\mathbb{K}$ is algebraically closed and that

$$[G_{\bar{a}'}:\pi(G_{\bar{a}}^0)]=[\pi(G_{\bar{a}}):\pi(G_{\bar{a}}^0)]\le[G_{\bar{a}}:G_{\bar{a}}^0]<\infty.$$

Hence the definable group $\pi(G_{\bar{a}}^0)$ has finite index in $G_{\bar{a}'}$. As $\left\{\frac{1}{a_0},\ldots,\frac{1}{a_{n-1}}\right\}$ is also linearly $\mathbb{F}_p$-independent, the group $G_{\bar{a}'}$ is connected by assumption. Therefore $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$.

Now, suppose that $G_{\bar{a}}$ is not connected.

Claim. *For every $\bar{x} \in G_{\bar{a}'}$, there exists a unique $x_n \in \mathbb{K}$ such that $(\bar{x}, x_n) \in G_{\bar{a}}^0$.*

Proof of the Claim. Assume there exists $\bar{x} \in \mathbb{K}^n$ and two distinct elements x_n^0 and x_n^1 of $\mathbb{K}$ such that $(\bar{x}, x_n^0)$ and $(\bar{x}, x_n^1)$ are elements of $G_{\bar{a}}^0$. As $G_{\bar{a}}^0$ is a group, their difference $(\bar{0}, x_n^0 - x_n^1)$ belongs also to $G_{\bar{a}}^0$. Thus, by definition of $G_{\bar{a}}$, its last coordinate $x_n^0 - x_n^1$ lies in $\mathbb{F}_p$. So $(\bar{0}, \mathbb{F}_p)$ is a subgroup of $G_{\bar{a}}^0$. Take an arbitrary element $(\bar{x}, x_n)$ in $G_{\bar{a}}$. As $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$, there exists $x_n' \in \mathbb{K}$ with $(\bar{x}, x_n') \in G_{\bar{a}}^0$. Again, the difference of the last coordinate $x_n' - x_n$ lies in $\mathbb{F}_p$. So

$$(\bar{x}, x_n) = (\bar{x}, x_n') - (\bar{0}, x_n' - x_n) \in G_{\bar{a}}^0.$$

This leads to a contradiction, as $G_{\bar{a}}^0$ is assumed to be a proper subgroup of $G_{\bar{a}}$. $\square_{\text{claim}}$

Thus, we can fix a definable additive function $f : G_{\bar{a}'} \to \mathbb{K}$ that sends every tuple to this unique element. Note that $G_{\bar{a}}$ and hence also $G_{\bar{a}}^0$ are defined over $\bar{a}$. So the function f is defined over $\bar{a}$ as well. Now, let $\bar{x} = (x_0, \ldots, x_{n-1})$ be any tuple in $G_{\bar{a}'}$ and set $L := \mathbb{F}_p(a_0, \ldots, a_n)$. Then:

$$x_n := f(\bar{x}) \in \mathrm{dcl}(\bar{a}, \bar{x}).$$

In other words, x_n is definable over $L(x_0, \ldots, x_{n-1})$ which simply means that it belongs to the purely inseparable closure $\bigcup_{n\in\mathcal{N}} L(x_0, \ldots, x_{n-1})^{p^{-n}}$ of $L(x_0, \ldots, x_{n-1})$ by [6, Chapter 4, Corollary 1.4]. Since there exists an $\ell \in L(x_0)$ such that $x_n^p - x_n - a_n^{-1}\ell = 0$, the element x_n is separable over $L(x_0, \ldots, x_{n-1})$. So it belongs to $L(x_0, \ldots, x_{n-1})$ which implies that there exists some mutually prime polynomials $g, h \in L[X_0, \ldots, X_{n-1}]$ such that

$$x_n = h(x_0, \ldots, x_{n-1})/g(x_0, \ldots, x_{n-1}).$$

Thus, by Lemma 7.6 the definable function $f(X_0, \ldots, X_{n-1})$ we started with is an additive polynomial in n variables over L^{alg} and there exists $c_{j,i}$ in L^{alg} and natural numbers m_j such that

$$f(X_0, \ldots, X_{n-1}) = \sum_{i=0}^{m_0} c_{0,i} X_0^{p^i} + \cdots + \sum_{i=0}^{m_{n-1}} c_{n-1,i} X_{n-1}^{p^i}.$$

Using the identities $X_i^p - X_i = \frac{a_0}{a_i}(X_0^p - X_0)$ in $G_{\bar{a}}^0$, there are β_j in L^{alg} and $g(X_0) = \sum_{i=1}^{m_0} d_i X_0^{p^i}$ an additive polynomial in $L^{\mathrm{alg}}[X_0]$ with summands of powers of X_0 greater or equal to p such that

$$f(X_0, \ldots, X_{n-1}) = g(X_0) + \sum_{j=0}^{n-1} \beta_j \cdot X_j.$$

Since the image under f of the vectors $(0,1,0,\ldots,0), (0,0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ has to be an element of $\mathbb{F}_p$, for $0 < i < n$ the β_i's have to be elements of $\mathbb{F}_p$. On the other hand, for any element $(x_0, \ldots, x_n)$ of $G_{\bar{a}}^0$ we have that $a_n(x_n^p - x_n) = a_0(x_0^p - x_0)$. Replacing x_n by

$f(x_0,\dots,x_{n-1})$ we obtain

$$\begin{aligned}0 &= a_n\left[f(x_0,\dots,x_{n-1})^p - f(x_0,\dots,x_{n-1})\right] - a_0(x_0^p - x_0)\\ &= a_n\left[g(x_0)^p - g(x_0) + (\beta_0^p x_0^p - \beta_0 x_0) + \sum_{j=1}^{n-1}\beta_j(x_j^p - x_j)\right] - a_0(x_0^p - x_0).\end{aligned}$$

Using again the identities $x_i^p - x_i = \frac{a_0}{a_i}(x_0^p - x_0)$ in $G_{\bar a}^0$ we obtain a polynomial in one variable

$$P(X) = a_n\left[g(X)^p - g(X) + (\beta_0^p X^p - \beta_0 X) + \sum_{j=1}^{n-1}\beta_j\frac{a_0}{a_j}(X^p - X)\right] - a_0(X^p - X)$$

which vanishes for all elements x_0 of $\mathbb{K}$ such that there exists $x_1,\dots,x_{n-1}$ in $\mathbb{K}$ with $(x_0,\dots,x_{n-1}) \in G_{\bar a'}$. In fact, this is true for all elements of $\mathbb{K}$. Hence, P is the zero polynomial. Notice that $g(X)$ appears in a pth-power. Since it contains only summands of power of X greater or equal to p, the polynomial $g(X)^p$ contains only summands of power of X strictly greater than p. As X only appears in powers less or equal to p in all summands of P except for $g(X)$, the polynomial $g(X)$ has to be the zero polynomial itself. By the same argument as for the other β_j, the coefficient β_0 has to belong to $\mathbb{F}_p$ as well. Dividing by a_0a_n yields that

$$\sum_{j=0}^{n}\beta_j\frac{1}{a_j}(X^p - X)$$

with $\beta_n := -1$ is the zero polynomial. Thus

$$\sum_{j=0}^{n}\beta_j\frac{1}{a_j} = 0.$$

As β_n is different from 0 and all β_i are elements of $\mathbb{F}_p$, this contradicts the assumption and the lemma is established. □

Using Lemma 7.7 and Fact 7.4, we obtain the following corollary in the same way as Kaplan, Scanlon and Wagner obtain [34, Corollary 2.9].

Corollary 7.8. *Let k be a perfect subfield of $\mathbb{K}$ and $\bar a \in k^n$ be as in the previous lemma. Then $G_{\bar a}$ is isomorphic over k to $(\mathbb{K},+)$. In particular, for any field $K \geq k$ with $K \leq \mathbb{K}$, the group $G_{\bar a}(K)$ is isomorphic to $(K,+)$.*

Proof. Consider the projection of $G_{\bar a}$ to its first coordinate. This map is onto and has finite fibers. Thus the dimension of $G_{\bar a}$ as a variety is equal to 1. As $G_{\bar a}$ is connected by Lemma 7.7, Fact 7.4 yields that $G_{\bar a}$ is isomorphic to $\mathbb{K}$ over k. Finally, this isomorphism send $G_{\bar a}(K)$ to K for any field that contains k. □

7.3 Artin-Schreier extensions

Definition 7.9. Let K be a field of characteristic $p > 0$ and $\wp(x)$ the additive homomorphism $x \mapsto x^p - x$. A field extension L/K is called an *Artin-Schreier extension* if $L = K(a)$ with $\wp(a) \in K$. We say that K is *Artin-Schreier closed* if it has no proper Artin-Schreier extension.

Observe, that if a is a root of the polynomial $\wp(x) - k$ for some k in K, then $\{a, a+1, \dots, a+p-1\}$ is the set of all roots of this polynomial. Hence for any a which is not in K such that $\wp(a)$ is in K, the field extension $K(a)$ is a cyclic Galois extension of degree p. Moreover, the converse holds as well, namely every cyclic Galois extension of degree p is an Artin-Schreier extension [42, Theorem VI.6.4].
Now, suppose that $K(a)$ and $K(b)$ are two Artin-Schreier extensions of K such that $\wp(a)$ and $\wp(b)$ lie in the same coset of the additive group K modulo $\wp(K)$, i. e. the element $\wp(a) - \wp(b)$ belongs to $\wp(K)$. Then we have that

$$\wp(a-b) = \wp(a) - \wp(b) \in \wp(K)$$

Thus there is $k \in K$ such that $\wp(a-b) = \wp(k)$. So the element $k-(a-b)$ is a root of $\wp(x)$ and thus it belongs to $\mathbb{F}_p$. As additionally k is an element of K, we have that $a-b$ belongs to K as well. Hence the two Artin-Schreier extensions $K(a)$ and $K(b)$ coincide. This demonstrates that the number of Artin-Schreier extensions is bounded by the cardinality of $K/\wp(K)$. Hence to show that K is Artin-Schreier closed, it suffices to show that K equals $\wp(K)$.

In the following remark, we produce elements from an algebraically independent array of size m^n which fit the condition of Lemma 7.7.

Remark 7.10. Let $\{\alpha_{i,j} : i \in n, j \in m\}$ be a set of algebraically independent elements in $\mathbb{K}$. Then the tuple $(a_{(i_0,\dots,i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ with $a_{(i_0,\dots,i_{n-1})} = \prod_{l=0}^{n-1} \alpha_{l,i_l}$ and ordered lexicographically satisfies the condition of Lemma 7.7.

Proof. Suppose that there exists a tuple of elements $(\beta_{(i_0,\dots,i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ in $\mathbb{F}_p$ not all equal to zero such that

$$\sum_{(i_0,\dots,i_{n-1}) \in m^n} \beta_{(i_0,\dots,i_{n-1})} \frac{1}{a_{(i_0,\dots,i_{n-1})}} = 0$$

Then the $\alpha_{i,j}$ satisfy:

$$\sum_{(i_0,\dots,i_{n-1}) \in m^n} \beta_{(i_0,\dots,i_{n-1})} \cdot \left(\prod_{\{(k,l) \neq (j,i_j) : j \leq n-1\}} \alpha_{k,l} \right) = 0$$

which contradicts the algebraic independence of the $\alpha_{i,j}$. □

We can now adapt the proof in [34] showing that an infinite dependent field is Artin-Schreier closed to n-dependent fields.

Theorem 7.11. *Any infinite n-dependent field is Artin-Schreier closed.*

Proof. Let K be an infinite n-dependent field. We may assume that it is $\aleph_0$-saturated. We work in a big algebraically closed field $\mathbb{K}$ that contains all objects we will consider. Let $k = \bigcap_{l\in\omega} K^{p^l}$, which is a type-definable infinite perfect subfield of K. We consider the formula $\psi(x; y_0, \dots, y_{n-1})$ given by $\exists t\ (x = \prod_{i=0}^{n-1} y_i \cdot \wp(t))$ which for every tuple $(a_0, \dots, a_{n-1})$ in k^n defines an additive subgroup of $(K, +)$. Let m be the natural number given by Proposition 2.6 for this formula. Now, we fix an array of size m^n of algebraically independent elements $\{\alpha_{i,j} : i \in n, j \in m\}$ in k and set $a_{(i_0,\dots,i_{n-1})}$ to be equal to $\prod_{l=0}^{n} \alpha_{l,i_l}$. By choice of m, there exists $(j_0, \dots, j_{n-1})$ in m^n such that

$$\bigcap_{(i_0,\dots,i_{n-1})\in m^n} a_{(i_0,\dots,i_{n-1})} \cdot \wp(K) = \bigcap_{(i_0,\dots,i_{n-1})\neq(j_0,\dots,j_{n-1})} a_{(i_0,\dots,i_{n-1})} \cdot \wp(K). \tag{7.1}$$

By reordering the elements, we may assume that $(j_0, \dots, j_{n-1})$ is equal to $(m, \dots, m)$. Let $\bar{a}$ be the tuple $(a_{(i_0,\dots,i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ ordered lexicographically and $\bar{a}'$ the restriction to $m^n - 1$ coordinates (one coordinate less).

We consider the groups $G_{\bar{a}}$ and respectively $G_{\bar{a}'}$ defined as in Definition 7.5. Using Remark 7.10 and Corollary 7.8 we obtain the following commuting diagram:

$$\begin{tikzcd} G_{\bar{a}} \arrow[r, "\pi"] \arrow[d, "\simeq"] & G_{\bar{a}'} \arrow[d, "\simeq"] \\ (\mathbb{K}, +) \arrow[r, "\rho"] & (\mathbb{K}, +) \end{tikzcd}$$

As the vertical isomorphisms are defined over k, this diagram can be restricted to K. Note that π and therefore also ρ stays onto for this restriction by equality (7.1) and that the size of $\ker(\rho)$ has to be equal to p. Choose a nontrivial element c in the kernel of ρ and let ρ' be equal to $\rho(c \cdot x)$. Observe that ρ' is still a morphism from $(\mathbb{K}, +)$ to $(\mathbb{K}, +)$, its restriction to K is still onto and its kernel is equal to $\mathbb{F}_p$. Then [34, Remark 4.2] ensures that ρ' is of the form $a \cdot (x^p - x)^{p^n}$ for some a in K. Finally, let $l \in K$ be arbitrary. Since $\rho' \restriction K$ is onto and X^{p^n} is an inseparable polynomial in characteristic p, there exists $h \in K$ with $l = h^p - h$. As $l \in K$ was arbitrary, we get that $\wp(K) = K$ and we can conclude. $\square$

The proof of [34, Corollary 4.4] adapts immediately and yields the following corollary.

Corollary 7.12. *If K is an infinite n-dependent field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

7.4 Non separably closed PAC field

The goal of this section is to generalize a result of Duret [16], namely that the theory of a non separably closed PAC field is not dependent. To do so we need the following two facts.

Fact 7.13. *[16, Lemme 6.2] Let K be a field and k be a subfield of K which is PAC. Let p be a prime number which does not coincide with the characteristic of K such that k contains all p^{th} roots of unity and there exists an element in k that does not have a pth root in K. Let*

$(a_i : i \in \omega)$ be a set of pairwise different elements of k and let I and J be finite disjoint subsets of ω. Then K realizes

$$\{\exists y(y^p = x + a_i) : i \in I\} \cup \{\neg\exists y(y^p = x + a_j) : j \in J\}.$$

Fact 7.14. *[16, Lemme 2.1] Every finite separable extension of a PAC field is PAC.*

Theorem 7.15. *Let K be a field and k be a subfield of K which is a non separably closed PAC field and relatively algebraically closed in K. Then* $\mathrm{Th}(K)$ *has the n-independence property.*

Proof. If k is countable, we may work in an elementary extension of the tuple (K,k) for which it is uncountable. As k is non separably closed, there exists a proper Galois extension l of k. Let p be a prime number that divides the degree of l over k. Then there is a separable extension k' of k such that the Galois extension l over k' is of degree p. We may distinguish two cases:

1. The characteristic of k is equal to p. As l is a cyclic Galois extension of degree p of k', a field of characteristic p, it is an Artin-Schreier extension of k'. We pick α such that $k' = k(\alpha)$ and let $K' = K(\alpha)$. As k' is separable over k, it is relatively algebraically closed in K' by [41, p.59]. Hence K' admits an Artin-Schreier extension and consequently its theory has IP_n by Theorem 7.11. As it is an algebraic extension of K, thus interpretable in K, the theory $\mathrm{Th}(K)$ has IP_n as well.

2. The characteristic of k is different than p. Since l is a separable extension of k', we can find an element β of l such that l is equal to $k'(\beta)$. Let ξ be a primitive p-root of unity and let $k'_\xi = k'(\xi)$ and $l_\xi = l(\xi)$. Note that l_ξ is equal to $k'_\xi(\beta)$, that the degree $[l_\xi : k'_\xi]$ is at most p, and that the degree $[k'_\xi : k']$ is strictly smaller than p. Additionally, we have:

$$[l_\xi : k'_\xi] \cdot [k'_\xi : k'] = [l_\xi : k'] = [l_\xi : l] \cdot [l : k'] = [l_\xi : l] \cdot p.$$

   Thus $[l_\xi : k'_\xi]$ is divisible by p and hence equal to p. Furthermore, the conjugates of β over k'_ξ are the same as over k'. Hence, as l is a Galois extension of k', they are contained in l, whence in l_ξ. Thus, the field l_ξ is a cyclic Galois extension of the field k'_ξ and k'_ξ contains the p-roots of unity. In other words, l_ξ is a Kummer extension of k'_ξ of degree p. So there exists an element δ in k'_ξ that does not have a p-root in it. Furthermore, as k'_ξ is a finite separable extension of k, it is also PAC by Fact 7.14 and it is relatively algebraically closed in $K'_\xi = K'(\xi)$ by [41, p.59]. Thus, the element δ has no p-root in K'_ξ as well. Let $\{a_{i,j} : j < n, i \in \omega\}$ be a set of algebraically independent elements in k'_ξ which exists as it is an uncountable field. This ensures that

$$\prod_{\ell=0}^{n-1} a_{i_\ell,\ell} \neq \prod_{\ell=0}^{n-1} a_{j_\ell,\ell} \quad \text{whenever} \quad (i_0,\dots,i_{n-1}) \neq (j_0,\dots,j_{n-1}).$$

   Thus we may apply Fact 7.13 to the fields K'_ξ, k'_ξ and the infinite set

$$\left\{\prod_{\ell=0}^{n-1} a_{i_\ell,\ell} : (i_0,\dots,i_{n-1}) \in \omega^n\right\}.$$

We deduce that for the formula $\varphi(y;x_0,\dots,x_{n-1})$ defined as $\exists z(z^p = y+\prod_{i=0}^{n-1} x_i)$ and for any disjoint finite subsets I and J of $\mathcal{N}^n$ there exists an element in K'_ξ that realizes

$$\{\varphi(y;a_{i_0,0},\dots,a_{i_{n-1},n-1})\}_{(i_0,\dots,i_{n-1})\in I} \cup \{\neg\varphi(y;a_{j_0,0},\dots,a_{j_{n-1},n-1})\}_{(j_0,\dots,j_{n-1})\in J}$$

Thus $\mathrm{Th}(K'_\xi)$ is n-dependent by compactness. As again K'_ξ is interpretable in K, we can conclude that the theory of K is n-dependent as well. □

Corollary 7.16. *The theory of any non separably closed PAC field has the IP_n property.*

In the special case of pseudofinite fields or, more generally, e-free PAC fields the previous corollary is a consequence of a result of Beyarslan proved in [4], namely that one can interpret the n-hypergraph in any such field.

7.5 Applications to valued fields

In [34] the authors deduce that an dependent valued field of positive characteristic p has to be p-divisible simply by the fact that infinite dependent fields are Artin-Schreier closed [34, Proposition 5.4]. Thus their result generalizes to our framework.

For the rest of the section, we fix some natural number n.

Corollary 7.17. *If (K,v) is an n-dependent valued field of positive characteristic p, then the value group of K is p-divisible.*

Together with Corollary 7.12, we can conclude the following analog to [34, Corollary 5.10].

Corollary 7.18. *Every n-dependent valued field of positive characteristic p whose residue field is perfect, is Kaplansky, i.e.*

- *the value group is p-divisible;*
- *the residue field is perfect and does not admit a finite separable extension whose degree is divisible by p.*

Now, we turn to the question whether an n-dependent henselian valued field can carry a nontrivial definable henselian valuation. Note that by a definable henselian valuation v on K we mean that the valuation ring of (K,v), i. e. the set of elements of K with non-negative value, is a definable set in the language of rings. We need the following definition:

Definition 7.19. Let K be a field. We say that its absolute Galois group is *universal* if for every finite group G there exist a finite extensions L of K and a Galois extension M of L such that $\mathrm{Gal}(M/L) \cong G$.

As any finite extension of an n-dependent field K of characteristic $p > 0$ is still n-dependent and of characteristic p, one cannot find finite extensions $L \subseteq M$ of K such that their Galois group $\mathrm{Gal}(M/L)$ is of order p. Hence any n-dependent field of positive characteristic has a non-universal absolute Galois group. Note that Jahnke and Koenigsmann showed in [32, Theorem 3.15] that a henselian valued field whose absolute value group is non universal and which is neither separably nor real closed admits a non-trivial definable henselian valuation. Hence this gives the following result which is a generalization of [32, Corollary 3.18]:

Proposition 7.20. *Let (K, v) be a non-trivially henselian valued field of positive characteristic p which is not separably closed. If K is n-dependent then K admits a non-trivial definable henselian valuation.*

8 Belastungstest für Divisionsringe

In this last chapter we study divisions rings. First, we analyze simple division rings which contain a generic element of weight 1 (Definition 8.12) and show that these are always commutative. Afterwards we move on to division rings of finite burden (Definition 8.18) which are shown to be finite dimensional over their center. This is joint work with Daniel Palacín.

8.1 Preliminaries

First, we summarize some general results on division rings we shall use.

Let us fix a division ring D. For any element a in D and any element d in $D^\times$, we write as usual a^d for $d^{-1}ad$. For any subdivision ring D_0 of D and every element $d \in D^\times$, we write D_0^d for $\{a^d : a \in D_0\}$. Moreover, for any a in D, we write a^D for its conjugacy class $\{a^d : d \in D^\times\}$ in D.

Fact 8.1 (Wedderburn's little theorem)**.** *[40, 13.1] Any finite division ring is a field.*

Fact 8.2 (Kaplansky's theorem)**.** *[40, 15.15] Any division ring that has finite exponent over its center is a field.*

Fact 8.3 (Cartan-Brauer-Hua Theorem)**.** *[40, 13.17] Let D be a division ring and D_0 be a subdivision ring. If for any d in $D^\times$, we have that D_0^d is contained in D_0, either D_0 is contained in the center of D, or D_0 equals D.*

Fact 8.4. *[40, 15.8] Let K be a commutative subdivision ring of D. If D has finite dimension over K, it is finite dimensional over its center.*

Now we turn to division rings in rosy and in particular simple theories. Note first the following:

Remark 8.5. Let D be a division ring and D_0 be a proper infinite subdivision ring of D. So D is a vector space over D_0 of dimension at least 2, and therefore the additive group $(D_0,+)$ has infinite index in $(D,+)$.

Remark 8.6. As the centralizer of an element is a subdivision ring and rosy groups satisfy the ICC^0 (Fact 2.3), by the previous remark any rosy division ring satisfies the ordinary ICC on centralizers.

We use the following fact due to Milliet on division ring of positive characteristic with a simple theory in the main result on simple division rings of weight 1:

Fact 8.7. *[45] A division ring of positive characteristic with a simple theory has finite dimension over its center.*

Remark 8.8. Analyzing the proof of Fact 8.7 one realizes that the only tools which are used are the chain condition on centralizers and Schlichting's theorem. Thus it can be generalized to rosy theories. Hence, any rosy division ring of positive characteristic has finite dimension over its center.

Now we introduce the notion of a generic element in groups with a simple theory and point out some properties.

Definition 8.9. Let G be a group $\emptyset$-definable in a simple theory and A be a parameter set. An element g of G is generic over A if for any element h of G with $h \perp_A g$ we have that $hg \perp A, h$. We say that a complete A-type is generic if all its realizations are.

One of the essential properties of generic elements is that they exist over any small parameter set in any group $\emptyset$-definable in a simple theory [63, Proposition 4.1.7]. Additionally, we make use of the following properties of generics elements and types which can be found in [63, Chapter 4]:

Properties 8.10. Let G be a group $\emptyset$-definable in a simple theory and A be a parameter set.

1. If g is a generic element of G over A and h is an element of G with $h \perp_A g$ then hg is generic over A as well.

2. Any A-definable subgroup H which contains an element of G which is generic over A has finite index in G.

8.2 Weight one

The following proposition will serve to show commutativity of simple division rings with a generic of weight 1. It uses ideas of the proof of [63, Theorem 5.6.12].

Proposition 8.11. *A division ring with a rosy theory which has finitely many non-central conjugacy classes is commutative.*

Proof. Let D be a non-commutative division ring with a rosy theory, and suppose that $a_0, \ldots, a_n$ are representatives of all its non-central conjugacy classes.

The first step is to show that $Z(D)$ is contained in $b^D - b$ for any non-central element b in D. To do so, we prove first that $Z(D) \cap (b^D - b)$ is an additive subgroup of $Z(D)$, with only finitely many $Z(D)$-translates:

Let $b^d - b$ and $b^c - b$ be two different elements in $(b^D - b) \cap Z(D)$. Note that their difference is an element of $Z(D)$. So, we may compute that:

$$(b^d - b) - (b^c - b) = (b^d - b^c)^{c^{-1}} = b^{dc^{-1}} - b \in (b^D - b).$$

So $(b^D - b) \cap Z(D)$ is an additive subgroup of $Z(D)$, which we denote by H. Now, for any z in $Z(D)$ we have that

$$zH = z[(b^D) - b] \cap Z(D) = [(zb^D) - zb] \cap Z(D).$$

Let c and c' be two element in $Z(D)$ such that $c'b$ is a conjugate of cb. Choose d in D such that $c'b = (cb)^d$. We have that

$$cH = (cH)^d = [(cb)^{Dd} - (cb)^d] \cap Z(D) = [(c'b)^D - (c'b)] \cap Z(D) = c'H.$$

As $Z(D)b$ contains only finitely many conjugacy classes, the group H has finitely many multiplicative $Z(D)$-translates.

Observe that for any two central elements z and z', their difference $z - z'$ belongs to $b^D - b$ if and only if there is some element x from $D^\times$ such that

$$b + z = b^x + z' = (b + z')^x.$$

As there are only finitely many conjugacy classes in $b + Z(D)$, the index of $Z(D) \cap (b^D - b)$ in $Z(D)$ has to be finite. Thus, the finite intersection of all its $Z(D)$-translates, which forms an ideal of $Z(D)$, has finite index in $Z(D)$ as well. If $Z(D)$ is finite, the characteristic of D is positive and thus by Remark 8.8 and Wedderburn's theorem (Fact 8.1), the division ring D must be commutative. So we may assume that $Z(D)$ is an infinite field and hence equal to $Z(D) \cap (b^D - b)$. Thus $Z(D)$ is contained in $b^D - b$ for any non-central b.

Now, by Kaplansky's theorem (Fact 8.2), we can find an element a in D for which none of its powers belong to $Z(D)$. As D satisfies the chain condition on centralizers, after replacing a by one of its powers, we may assume that $C_D(a) = C_D(a^n)$ for any natural number n.

Suppose first that there exists a natural number n and an element c in $Z(D)$ with no n-root in D. As a^n is non-central, there is $x \in D$ such that $(a^n)^x - a^n = c$. Observe that a and a^x commute since

$$C_D(a^x) = C_D(a)^x = C_D(a^n)^x = C_D((a^n)^x) = C_D(a^n) = C_D(a)$$

and so

$$(a^x a^{-1})^n - ca^{-n} = ((a^n)^x - c)a^{-n} = 1.$$

However, as ca^{-n} is non-central, one can find an element y in D with $(ca^{-n})^y - ca^{-n} = 1$ and so the n-power $(a^x a^{-1})^n$ equals to $(ca^{-n})^y$. As c was assumed to have no n-root in D, this yields a contradiction.

Otherwise, for any natural number m any element of the center has an m^{th}-root in D. In particular, there is an infinite sequence $\xi_0, \xi_1, \xi_2, \ldots$ of elements in D with $\xi_k^{2^k} = -1$ for all $k < \omega$. It is clear that all these roots of unity have different conjugacy classes and hence all but finitely many must belong to the center since there are only finitely many non-central conjugacy classes. So let I be the set of indices such that ξ_i belongs to $Z(D)$. So $\{\xi_i a : i \in I\}$ is a sequence of non-central elements. As again there are only finitely many non-central conjugacy classes, one can find two different indices i and j

in I and some x in $D \setminus C_D(a)$ such that $\xi_i a = \xi_j a^x$. Thus, $a = \zeta a^x$ with $\zeta^m = 1$ for some $m < \omega$ and $\zeta \in Z(D)$. Hence $a^m = (\zeta a^x)^m = (a^m)^x$ and so x belongs to the centralizer of a^m which, by the choice of a, coincides with the centralizer of a. This yields the final contradiction. □

Now, we introduce the notion of the weight of a type and point out the properties which we use in our proof.

Definition 8.12. Let p be a type and λ be a cardinal. The *weight* of p is at least λ if there is a non-forking extension $\mathrm{tp}(a/A)$ of p and a sequence $(a_i : i < \lambda)$ such that for all $i < \lambda$:

- $a_i \mathop{\smile\mkern-12mu\mid}_A (a_j : j < i)$;
- $a \not\mathop{\smile\mkern-12mu\mid}_A a_i$.

The weight of p is λ, denoted by $w(p) = \lambda$, if it is at least λ and not at least λ^+.
For an element g of G and a parameter set A, we write $w(g/A)$ for $w(\mathrm{tp}(g/A))$.

Properties 8.13. Let G be a group $\emptyset$-definable in a simple theory, A and B be parameter sets and g be an element of G.

- A type p over A having weight 1 implies that for every realization a of p and any two elements b and c such that $b \mathop{\smile\mkern-12mu\mid}_A c$, we have that $a \mathop{\smile\mkern-12mu\mid}_A b$ or $a \mathop{\smile\mkern-12mu\mid}_A c$.
- [63, Lemma 5.2.2] If $g \mathop{\smile\mkern-12mu\mid}_A B$, then $w(g/A) = w(g/A, B)$.
- Let h be an element of G inter-algebraic with g over A, then $w(g/A) = w(h/A)$.

Remark 8.14. Let G be a group with a simple theory and p and q be two generic types over A. Then $w(p) = w(q)$.

Proof. Let g be a realization of p and h be a realization of q such that $g \mathop{\smile\mkern-12mu\mid}_A h$. Thus

$$gh^{-1} \mathop{\smile\mkern-12mu\mid}_A g \quad \text{and} \quad gh^{-1} \mathop{\smile\mkern-12mu\mid}_A h.$$

Thus

$$w(p) = w(g/A) = w(g/A, gh^{-1}) = w(h/A) = w(q).$$

□

Krupinski and Pillay show in [39, Remark 1.1] that the set of non-generic elements of any stable group whose generic types have weight 1 forms a subgroup. The proof is easily adaptable for simple theories. For sake of completeness we give a detailed proof.

Lemma 8.15. *Let G be a group $\emptyset$-definable in a simple theory for which one generic type (and thus all) has weight 1. Then, the set of non-generic elements of G over any small parameter set A forms a subgroup.*

Proof. Let a and b be two non-generic elements and suppose towards a contradiction that ab is generic. Now, choose a generic element g of G over A independent of a, b over A. As g and ab are generic and $g \mathop{\smile\!\!\!\!\!\!\,|}_A b$ as well as $g \mathop{\smile\!\!\!\!\!\!\,|}_A ab$ we obtain that bg and abg are generic over A and $g \mathop{\smile\!\!\!\!\!\!\,|}_A abg$.

Now suppose that $bg \mathop{\smile\!\!\!\!\!\!\,|}_A g$. This yields that g is generic over A, bg, which implies that $bgg^{-1} = b$ is generic which contradicts our assumption. Hence

$$bg \not\mathop{\smile\!\!\!\!\!\!\,|}_A g.$$

On the other hand using the same argument, we obtain that $bg \not\mathop{\smile\!\!\!\!\!\!\,|}_A abg$ as a is non-generic. As $g \mathop{\smile\!\!\!\!\!\!\,|}_A abg$, this implies that the weight of $\mathrm{tp}(bg/A)$ is at least 2. On the other hand, as gb is a generic element over A, it has weight 1 which leads to a contradiction and the lemma is established. □

As multiplicative and additive generics in a division ring with a simple theory coincide, we obtain immediately the following corollary.

Corollary 8.16. *The set of non-generic elements over any given small set of parameters of any division ring with a simple theory with a generic of weight* 1 *forms a subdivision ring.*

Theorem 8.17. *A definable division ring in a simple theory with a generic element of weight* 1 *is a field.*

Proof. Suppose that D is a non-commutative division ring with a simple theory and a generic element of weight 1. Let g be any non-central element. We denote by $\ulcorner g^D \urcorner$ the canonical parameter of the conjugacy class of g in D. Now, let X be the set of non-generic elements of D over $\ulcorner g^D \urcorner$. By Corollary 8.16 the set of non-generic elements over any given small subset forms a division ring. As conjugation is an automorphism of D which fixes $\ulcorner g^D \urcorner$, such a subdivision ring is invariant under conjugation. Thus, as it is properly contained in D, we have that the division ring of non-generics over $\ulcorner g^D \urcorner$ is contained in $Z(D)$ by the Cartan-Brauer-Hua Theorem (Fact 8.3). In fact, as $Z(D)$ is a $\emptyset$-definable proper subdivision ring, it has infinite index as additive subgroup. Thus, it cannot contain any generic element and therefore the division ring of non-generics over $\ulcorner g^D \urcorner$ and $Z(D)$ coincide. So g itself is a generic element of D independent of $\ulcorner g^D \urcorner$. Thus for any noncentral element g in D, we have that $\ulcorner g^D \urcorner$ is algebraic over the empty set. Hence D has only finitely many non-central conjugacy classes, whence it is commutative by Proposition 8.11. □

8.3 Finite burden

In this section we want to analyze division ring whose theory has finite burden. This is a subclass of NTP_2 theories. Moreover, the burden of a complete type in a simple theory is the supremum of the weights of all its extensions. Below we give the precise definition.

Definition 8.18. Let $p(x)$ be a (partial) type. An *inp-pattern of depth* κ in $p(x)$ is a sequence of formulas $(\psi_\alpha(\bar{x};\bar{y}_\alpha) : \alpha < \kappa)$, an array of parameters $(\bar{a}_{\alpha,j} : \alpha < \kappa, j < \omega)$ with $|\bar{a}_{\alpha,j}| = |\bar{y}_\alpha|$, and a sequence of natural numbers $(k_\alpha : \alpha < \kappa)$ such that:

- $\{\psi_\alpha(\bar{x};\bar{a}_{\alpha,j}) : j \in \omega\}$ is k_α-inconsistent for every $\alpha < \kappa$;
- $p(x) \cup \{\psi_\alpha(\bar{x};\bar{a}_{\alpha,f(\alpha)}) : \alpha \in \kappa\}$ is consistent for every $f : \kappa \to \omega$.

A theory has *burden* n for some natural number n, if there is no inp-pattern of depth n in the partial type $x = x$. A theory of burden 1 is called *inp-minimal*.
A definable group or division ring has burden n if the formula which defines the group or division ring seen as a partial type has burden at most n.

The following result corresponds to [12, Proposition 4.5] in the definable context. We offer a proof for the sake of completeness.

Lemma 8.19. *Let G be a definable group of burden n and let $H_0,\ldots,H_n$ be definable normal subgroups of G. Then there exists some $j \le n$ such that $\bigcap_i H_i$ has finite index in $\bigcap_{i\neq j} H_i$.*

Proof. Suppose, towards a contradiction, that there are definable normal subgroups $H_0,\ldots,H_n$ of G such that the intersection of all of them, denoted by N, has infinite index in each proper subintersection $H_{\neq i}$ defined by $\bigcap_{j\neq i} H_j$. Let $\{a_i^j\}_{i\in\omega}$ be a collection of representatives of distinct cosets of N in $H_{\neq j}$. Thus, the family $\{a_i^j H_j\}_{i\in\omega}$ for a fixed $j \le n$ consists of pairwise disjoint cosets of H_j in G and is therefore 2-inconsistent. On the other hand, the intersection $a_{i_0}^0 H_0 \cap \ldots \cap a_{i_n}^n H_n$ is nonempty for any choice of $i_0,\ldots,i_n \in \omega$ as each H_i is normal in G. Hence, the formulas defining cosets of the H_j's together with the a_i^j's contradict the fact that G has burden n. □

Theorem 8.20. *A division ring of burden n has dimension at most n over any infinite definable subfield.*

Proof. Let D be a division ring of burden n with an infinite definable subfield K, and assume that the dimension of D over K is at least $n+1$. Choose K-linearly independent elements $e_0,\ldots,e_n$ in D. For $j \le n$, consider the definable K-vector spaces $V_j = \bigoplus_{i\neq j}\langle e_i\rangle$, and observe that all of them are normal subgroups in D^+, as the latter is abelian. Moreover, as the $e_0,\ldots,e_n$ are linearly independent elements and each V_j is generated by all these elements but the element e_j, we have that $\bigcap_j V_j$ is the zero vector space and for some ℓ less or equal to n, the vector space $\bigcap_{j\neq\ell} V_j$ is generated by e_ℓ. Therefore, Lemma 8.19 yields the existence of some $k \le n$ such that the index

$$\left[\bigcap_{j\neq k} V_j : \bigcap_j V_j\right] = [\langle e_k\rangle : \{0\}]$$

is finite. However, as K is infinite and $\langle e_k\rangle$ is a one-dimensional K-vector space, we obtain the desired contradiction. □

Corollary 8.21. *Any infinite division ring of burden n has dimension at most n over its center.*

Proof. Let D be a division ring of burden n. As any division ring of finite order is commutative by Kaplansky's theorem (Fact 8.2), we may assume that D has an element d of infinite order. Hence $Z(C(d))$ is an infinite subfield of D and so Theorem 8.20 implies that D has finite dimension over $Z(C(d))$. Thus the ring D has finite dimension over its center by Fact 8.4 which must hence be infinite. Now, we can apply Theorem 8.20 to the center of D and obtain the desired result. □

Immediately we obtain:

Corollary 8.22. *An inp-minimal division ring is commutative.*

Moreover, as the quaternions are a finite extension of the inp-minimal field $\mathbb{R}$, they have finite burden. As they are non-commutative, one cannot expects the improve Theorem 8.20 to obtain commutativity.

Another consequence of Theorem 8.20 is a descending chain condition among definable subfields.

Corollary 8.23. *Let D be an infinite division ring of burden n. Then any descending chain of definable infinite subfields has length at most $\lfloor \log_2(n) \rfloor + 1$. Therefore, if $\mathcal{F}$ is a family of definable subfields of D, the intersection of all subfields in $\mathcal{F}$ is equal to a finite subintersection and so, it is definable.*

Proof. Assume that D has burden n and suppose, towards a contradiction, that there exists a proper descending chain

$$D = F_0 \gneq F_1 \gneq F_2 \gneq \cdots \gneq F_{\lfloor \log_2(n) \rfloor + 1}$$

of infinite subfields of D. Hence, the dimension of F_{i+1} over F_i as a vector space is greater or equal to 2. This implies that dimension of F_0 over $F_{\lfloor \log_2(n) \rfloor + 1}$ is at least $2^{\lfloor \log_2(n) \rfloor + 1}$, which contradicts Theorem 8.20.

For the second part of the statement note that we may suppose any finite intersection of subfields in $\mathcal{F}$ to be infinite (otherwise the result is obvious). So the result follows from the first part. □

Now, we consider the two-sorted structure of an infinite field F and a subgroup G of $\mathrm{Aut}(F)$ equipped with the natural action of G on F, i.e. the structure

$$(F, G, +_F, \times_F, \text{action of } G \text{ on } F).$$

In the superstable case, this setting has been already analyzed by Hrushovski in [29], who obtained the following:

Fact 8.24. *[29, Proposition 3] If the structure $(F, G, +_F, \times_F, \text{action of } G \text{ on } F)$ is superstable, then G is trivial.*

Our aim is to generalize this result to the finite burden framework. In particular, the same holds in the inp-minimal case.

Theorem 8.25. *If the structure $(F, G, +_F, \times_F, \text{action of } G \text{ on } F)$ has burden n and the algebraic closure of the prime field of F in F is infinite, then G has size at most n.*

Proof. Assume, as we may, that our structure is sufficiently saturated. Let k be the prime field of F and let G_x denote the stabilizer of any element $x \in k^{\mathrm{alg}} \cap F$ in G. As k is fixed by the action of every element in G and G/G_x is in bijection with the orbit of x, the stabilizer G_x has finite index in G. Now, we work with the subgroup

$$H = \bigcap_{x \in k^{\mathrm{alg}} \cap F} G_x$$

of G. Note that it is a type-definable subgroup of G of bounded index. We consider the intersection $\mathrm{Fix}(H) = \bigcap_{\sigma \in H} \mathrm{Fix}(\sigma)$ of definable subfields of F. By Corollary 8.23 it is equal to a finite subintersection. Hence, as additionally $\mathrm{Fix}(H)$ contains the infinite field $k^{\mathrm{alg}} \cap F$, it is a definable infinite subfield of F. Thus, Theorem 8.20 yields that F has at most dimension n over $\mathrm{Fix}(H)$, so H is finite (by Galois theory). Hence the group G is a bounded definable group and whence finite by compactness. Now, consider the definable field $\mathrm{Fix}(G)$. By Galois theory we know that F is a finite field extension of $\mathrm{Fix}(G)$ of degree $|G|$, and so $\mathrm{Fix}(G)$ is an infinite definable subfield of F. Hence F has dimension at most n over $\mathrm{Fix}(G)$ by Theorem 8.20 and whence G has size at most n. □

Corollary 8.26. *If the structure $(F, G, +_F, \times_F$, action of G on $F)$ is inp-minimal and the algebraic closure of the prime field of F in F is infinite, then G is trivial.*

Bibliography

[1] T. Altinel and P. Baginski. Definable envelopes of nilpotent subgroups of groups with chain conditions on centralizers. *Proc. Amer. Math. Soc.*, 142(5):1497–1506, 2014.

[2] R. Baer. Endlichkeitskriterien für Kommutatorgruppen. *Math. Ann.*, 124:161–177, 1952.

[3] J. T. Baldwin and J. Saxl. Logical stability in group theory. *J. Austral. Math. Soc. Ser. A*, 21(3):267–276, 1976.

[4] Ö. Beyarslan. Random hypergraphs in pseudofinite fields. *J. Inst. Math. Jussieu*, 9(1):29–47, 2010.

[5] T. Blossier. Subgroups of the additive group of a separably closed field. *Ann. Pure Appl. Logic*, 134(2-3):169–216, 2005.

[6] E. Bouscaren, editor. *Model theory and algebraic geometry*, volume 1696 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture.

[7] R. M. Bryant. Groups with the minimal condition on centralizers. *J. Algebra*, 60(2):371–383, 1979.

[8] G. Cherlin and S. Shelah. Superstable fields and groups. *Ann. Math. Logic*, 18(3):227–270, 1980.

[9] A. Chernikov. *On theories without the tree property of the second kind*. PhD thesis, Universié Claude Bernard Lyon 1, 2012.

[10] A. Chernikov and M. Hils. Valued difference fields and NTP_2. *Israel J. Math.*, 204(1):299–327, 2014.

[11] A. Chernikov and I. Kaplan. Forking and dividing in NTP_2 theories. *J. Symbolic Logic*, 77(1):1–20, 2012.

[12] A. Chernikov, I. Kaplan, and P. Simon. Groups and fields with NTP_2. *Proc. Amer. Math. Soc.*, 143(1):395–406, 2015.

[13] A. Chernikov, D. Palacín, and K. Takeuchi. On n-dependence. *submitted*, arXiv:1411.0120, 2014.

[14] R. de Aldama. Definable nilpotent and soluble envelopes in groups without the independence property. *MLQ Math. Log. Q.*, 59(3):201–205, 2013.

[15] J. Derakhshan and F. O. Wagner. Nilpotency in groups with chain conditions. *Quart. J. Math. Oxford Ser. (2)*, 48(192):453–466, 1997.

[16] J.-L. Duret. Les corps faiblement algébriquement clos non séparablement clos ont la propriété d'indépendence. In *Model theory of algebra and arithmetic (Proc. Conf., Karpacz, 1979)*, volume 834 of *Lecture Notes in Math.*, pages 136–162. Springer, Berlin-New York, 1980.

[17] U. Felgner. On $\aleph_0$-categorical extra-special p-groups. *Logique et Analyse (N.S.)*, 18(71-72):407–428, 1975. Comptes Rendus de la Semaine d'Étude en Théorie des Modèles (Inst. Math., Univ. Catholique Louvain, Louvain-la-Neuve, 1975).

[18] H. Fitting. Beiträge zur Theorie der Gruppen endlicher Ordnung. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 48:77–141, 1938.

[19] D. Goss. *Basic structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1996.

[20] F. Haimo. The FC-chain of a group. *Canadian J. Math.*, 5:498–511, 1953.

[21] P. Hall. Some sufficient conditions for a group to be nilpotent. *Illinois J. Math.*, 2:787–801, 1958.

[22] B. Hart, E. Hrushovski, and M. C. Laskowski. The uncountable spectra of countable theories. *Ann. of Math. (2)*, 152(1):207–257, 2000.

[23] N. Hempel. On n-dependent groups and fields. *accepted for publication in MLQ Math. Log. Q. , arXiv:1401.4880*, 2015.

[24] N. Hempel and A. Onshuus. Groups in NTP_2. *accepted for publication in Israel J. Math. , arXiv:1510.01049*, 2015.

[25] C. W. Henson. A family of countable homogeneous graphs. *Pacific J. Math.*, 38:69–83, 1971.

[26] K. K. Hickin and J. A. Wenzel. On the normal product of FC-nilpotent and FC-hypercentral groups. *Acta Math. Acad. Sci. Hungar.*, 22:355–358, 1971/72.

[27] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[28] A. O. Houcine. A remark on the definability of the Fitting subgroup and the soluble radical. *MLQ Math. Log. Q.*, 59:62–65, 2013.

[29] E. Hrushovski. On superstable fields with automorphisms. In *The model theory of groups (Notre Dame, IN, 1985–1987)*, volume 11 of *Notre Dame Math. Lectures*, pages 186–191. Univ. Notre Dame Press, Notre Dame, IN, 1989.

[30] E. Hrushovski. Pseudo-finite fields and related structures. In *Model theory and applications*, volume 11 of *Quad. Mat.*, pages 151–212. Aracne, Rome, 2002.

[31] B. Huppert. *Endliche Gruppen. I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[32] F. Jahnke and J. Koenigsmann. Definable Henselian valuations. *J. Symb. Log.*, 80(1):85–99, 2015.

[33] T. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin Heidelberg, third edition, 2003.

[34] I. Kaplan, T. Scanlon, and F. O. Wagner. Artin-Schreier extensions in NIP and simple fields. *Israel J. Math.*, 185:141–153, 2011.

[35] B. Kim. Forking in simple unstable theories. *J. London Math. Soc. (2)*, 57(2):257–267, 1998.

[36] B. Kim. Simplicity, and stability in there. *J. Symbolic Logic*, 66(2):822–836, 2001.

[37] B. Kim and A. Pillay. Simple theories. *Ann. Pure Appl. Logic*, 88(2-3):149–164, 1997. Joint AILA-KGS Model Theory Meeting (Florence, 1995).

[38] B. Kim and A. Pillay. From stability to simplicity. *Bull. Symbolic Logic*, 4(1):17–36, 1998.

[39] K. Krupiński and A. Pillay. On stable fields and weight. *J. Inst. Math. Jussieu*, 10(2):349–358, 2011.

[40] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[41] S. Lang. *Introduction to algebraic geometry*. Interscience Publishers, Inc., New York-London, 1958.

[42] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[43] M. C. Laskowski. Vapnik-Chervonenkis classes of definable sets. *J. London Math. Soc. (2)*, 45(2):377–384, 1992.

[44] A. Macintyre. On ω_1-categorical theories of fields. *Fund. Math.*, 71(1):1–25. (errata insert), 1971.

[45] C. Milliet. Stable division rings. *J. Symbolic Logic*, 76(1):348–352, 2011.

[46] C. Milliet. Definable envelopes in groups having a simple theory. *accepted for publication in J. Algebra*, *HAL-00657716*, 2012.

[47] C. Milliet. On properties of (weakly) small groups. *J. Symbolic Logic*, 77(1):94–110, 2012.

[48] S. Montenegro. Pseudo real closed field, pseudo p-adic closed fields and NTP_2. *preprint*, arXiv:1411.7654, 2014.

[49] B. H. Neumann. Groups covered by permutable subsets. *J. London Math. Soc.*, 29:236–248, 1954.

[50] D. Palacín and F. O. Wagner. A Fitting theorem for simple groups. *accepted for publication in Bull. London Math. Soc.*, *arXiv:1410.2583*, 2014.

[51] A. Pillay and B. Poizat. Corps et chirurgie. *J. Symbolic Logic*, 60(2):528–533, 1995.

[52] B. Poizat. *Stable groups*, volume 87 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1987 French original by Moses Gabriel Klein.

[53] M. Rosenlicht. On a result of Baer. *Proc. Amer. Math. Soc.*, 13:99–101, 1962.

[54] T. Scanlon. Infinite stable fields are Artin-Schreier closed. *notes*, *http://math.berkeley.edu/ scanlon/papers/ASclos.pdf*, 1999.

[55] G. Schlichting. Operationen mit periodischen Stabilisatoren. *Arch. Math. (Basel)*, 34(2):97–99, 1980.

[56] S. Shelah. Stable theories. *Israel J. Math.*, 7:187–202, 1969.

[57] S. Shelah. Simple unstable theories. *Ann. Math. Logic*, 19(3):177–203, 1980.

[58] S. Shelah. *Classification theory for abstract elementary classes. Vol. 2*, volume 20 of *Studies in Logic (London)*. College Publications, London, 2009. Mathematical Logic and Foundations.

[59] S. Shelah. Dependent first order theories, continued. *Israel J. Math.*, 173:1–60, 2009.

[60] S. Shelah. Definable groups for dependent and 2-dependent theories. *arXiv:math/0703045*, 2012.

[61] P. Simon. *A guide to NIP theories*, volume 44 of *Lecture Notes in Logic*. ASL and Cambridge University Press, Berlin, 2015.

[62] F. O. Wagner. The Fitting subgroup of a stable group. *J. Algebra*, 174(2):599–609, 1995.

[63] F. O. Wagner. *Simple theories*, volume 503 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[64] J. S. Wilson. On simple pseudofinite groups. *J. London Math. Soc. (2)*, 51(3):471–490, 1995.

Groups and Fields in Neostable Theories

Chain Conditions and Definable Envelopes

Abstract: This thesis is dedicated to the study of groups and fields whose definable sets do not admit certain combinatorial patterns.
Given a group G, one particular problem we are interested in is to find definable envelopes for arbitrary abelian, nilpotent or solvable subgroups of G which admit the same algebraic properties. Such evelopes exists if G is stable and even if G is merely dependent but sufficiently saturated, with the additional hypothesis of normality in the solvable case. In groups with a simple theory, one obtains definable envelopes *up to finite index*.
We introduce the notion of an almost centralizer and establish some of its basic properties. This enables us to extend the aforementioned results to $\widetilde{\mathfrak{M}}_c$-groups, i. e. groups in which any definable section satisfies a chain condition on centralizers up to finite index. These include any definable group in a rosy and in particular in a simple theory. Furthermore, inspired from the proof in dependent theories as well as using techniques developed for almost centralizers in this thesis, we are able to find definable envelopes up to finite index for abelian, nilpotent and normal solvable subgroups of any enough saturated NTP_2 group. Moreover, using envelopes for nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups and the chain condition on centralizer up to finite index, we show additionally that the Fitting subgroup of any $\widetilde{\mathfrak{M}}_c$-group is nilpotent and that its almost Fitting subgroup is virtually solvable.
The second part of this thesis focuses on the study of n-dependent fields. We prove that any n-dependent field is Artin-Schreier closed and that non separably closed PAC fields are not n-dependent for any natural number n.

Keywords: definable envelopes, NTP_2 groups, chain condition on centralizers, almost centralizers, n-dependent theories

Groupes et Corps dans des Théories Neostables

Condition de Chaîne et Enveloppes Définissables

Résumé: Cette thèse est consacrée à l'étude des groupes et des corps dont les ensembles définissables n'admettent pas certaines configurations combinatoires.
Étant donné un groupe G, un problème particulier qui nous intéresse est de trouver des enveloppes définissables de sous-groupes abéliens, nilpotents ou résolubles de G ayant les mêmes propriétés algébriques. De tels enveloppes existent si G est stable, et même si G est seulement dépendant mais saturé, avec l'hypothèse supplémentaire de normalité pour le cas des sous-groupes résolubles. Dans les groupes ayant une théorie simple, on obtient des enveloppes définissables *à indice fini près*.
Nous introduisons la notion de presque centralisateur et nous établissons certaines de ses propriétés de base. Cela nous permet d'étendre les résultats mentionnés ci-dessus à des $\widetilde{\mathfrak{M}}_c$-groupes, i. e. des groupes dans lesquels toutes sections définissables satisfont une condition de chaîne sur les centralisateurs à indice fini près. Ceux-ci incluent les groupes définissables dans une théorie rose et en particulier dans une théorie simple. En s'inspirant de la preuve pour les groupes dépendants et en utilisant les techniques développées sur les presque centralisateurs dans cette thèse, nous démontrons l'existence des enveloppes définissables à indice fini près pour des sous groupes abélien, nilpotents ou normaux et résolubles de tout groupe NTP_2 assez saturé. En utilisant les enveloppes des sous-groupes nilpotents de $\widetilde{\mathfrak{M}}_c$-groupes et la condition de chaîne sur les centralisateurs à indice fini près, nous montrons en outre que le sous-groupe de Fitting de tout $\widetilde{\mathfrak{M}}_c$-groupe est nilpotent et que son sous-groupe presque Fitting est résoluble-par-fini.
La deuxième partie de cette thèse porte sur l'étude des corps n-dépendants. Nous démontrons que tout corps n-dépendant est Artin-Schreier clos et que les corps PAC non séparablement clos ne sont pas n-dépendants pour tout nombre naturel n.

Mots clés: enveloppes définissable, groupes NTP_2, condition de chaîne sur les centralisateur, presque centralisateur, théorie n-dépendante

Image en couverture : Forking tree

ED 512

INFOMATHS

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